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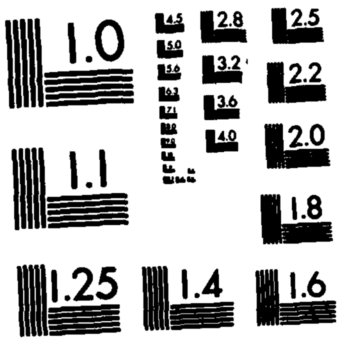

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MULTIMODEL DESIGN OF LARGE SCALE SYSTEMS WITH MULTIPLE [illegible]

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The central theme of this thesis is multimodeling. It is concerned with modeling and control strategy interaction in a multimodel context. Realistic situations are studied, which allow the decision makers to use different simplified models of the system. Three different approaches to multimodeling are examined. Firstly, within the framework of multiparameter singular perturbations, we demonstrate the well-posedness of an a-priori selected multimodeling scheme, for a class of Nash and team problems. This establishes,

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MULTIMODEL DESIGN OF LARGE SCALE SYSTEMS WITH MULTIPLE DECISION MAKERS

by

Vikram Raj Saksena

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MULTIMODEL DESIGN OF LARGE SCALE SYSTEMS WITH MULTIPLE DECISION MAKERS

BY

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Electrical Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 1982

Thesis Adviser: Professor J. B. Cruz, Jr.

Urbana, Illinois

To

My Parents

and

My Wife

असतोमा सद्गमय

तमसोमा ज्योतिर्गमय

मृत्योर्मा अमृतंगमय

Lead me from evil to virtue.
Lead me from darkness to light.
Lead me from death to eternal life.

(Vedic Prayer)

MULTIMODEL DESIGN OF LARGE SCALE SYSTEMS WITH MULTIPLE DECISION MAKERS

Vikram Raj Saksena, Ph.D.
Department of Electrical Engineering
University of Illinois at Urbana-Champaign, 1982

The central theme of this thesis is multimodeling. It is concerned with modeling and control strategy interaction in a multimodel context. Realistic situations are studied, which allow the decision makers to use different simplified models of the system. Three different approaches to multimodeling are examined. Firstly, within the framework of multiparameter singular perturbations, we demonstrate the well-posedness of an a-priori selected multimodeling scheme, for a class of Nash and team problems. This establishes, in some sense, the "robustness" of this multimodeling scheme to a class of solution concepts and information patterns. Secondly, for a class of weakly-coupled Markov chains, we use a perturbational approach to develop an efficient algorithm for computing near-optimal incentive policies, which allows for multimodeling on the part of the decision makers. Finally, for a class of linear-quadratic problems, we use an input-output approach to restructure the problem, and choose appropriate admissible strategies which induce multimodel solutions.

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CHAPTER 1

INTRODUCTION

The problem of efficient management and control of large scale systems has been extremely challenging to control engineers. There are essentially two main issues of concern. The modeling issue is complicated due to the large dimension of the system. The crucial problem here is one of model simplification, i.e., how to obtain a simplified low-order model of the system which would result in an acceptable control design [2,4,5]. In large scale systems the model simplification problem is intimately related to notions of time-scales, weak-coupling and controllability-observability [1-8]. The control design issue is complicated due to the presence of multiple decision makers having possibly different goals and possessing decentralized information. The crucial problem here is to obtain optimal multicontroller strategies under nonclassical information patterns and various cooperative and noncooperative solution concepts [9-13]. In large scale system design the intricate relationship between the modeling and strategy design issues introduces additional complexities not encountered while considering each problem in isolation. This is due to the fact that many aspects of the system structure are variant under the control actions. Many cases of ill-posed closed-loop designs based on reduced-order models have been reported (see for example [45]). The complexities get more involved when there are multiple decision makers as opposed to a centralized decision maker [21-24]. This is because each decision maker's perception of the system structure and dynamics may be altered by the actions of the other decision makers. Hence any approach towards developing an efficient design

methodology must treat the modeling and strategy design issues in a unified framework.

The central theme of this thesis is multimodeling. It is concerned with modeling and control strategy interaction in a multimodel context. In large scale system design, it is desirable to allow the decision makers to use different simplified models of the system [63], due to: i) the necessity to ease the computational burden associated with simulation, analysis, and design; ii) the need to obtain a simplified control structure which is feasible to implement; and iii) a lack of adequately modeled dynamics of some parts of the system. In this thesis we study realistic situations, which allow the decision makers to use different models of the system. It is our purpose to strengthen and extend the multimodeling concept beyond the framework within which it was originally introduced in [14,15]. Towards this end, we examine three different approaches to multimodeling. Firstly, we consider situations, when a rational choice of the multimodeling scheme is made a-priori, based solely on the model structure. To establish the validity of such a scheme we then examine its impact on the design of control strategies. Specifically, our two main issues of concern are: the preservation of stability; and, a minimal loss in performance. Secondly, we explore multimodeling possibilities in numerical algorithms which compute near-optimal policies. Finally, we attempt to induce multimodel solutions by an appropriate re-structuring of the problem, and a suitable choice of admissible strategies. We hope our study would reveal the interplay between the structural features of the system like time-scales, weak-coupling, controllability-observability, and strategy design under nonclassical information patterns; and help us to achieve a better understanding of the multimodeling concept.

The concept of multimodel strategies for large scale systems has been introduced in [14,15] within the framework of multiparameter singular perturbations. In this framework, a large scale system is viewed as consisting of a "slow" core coupled to a number of "fast" subsystems. A multimodel situation results when each decision maker models the dynamics of one fast subsystem and assumes a certain reduced-order equivalent of the rest of the system. The design objective of each decision maker is assumed to be compatible with the multimodel assumptions, i.e., each decision maker is assumed not to penalize the neglected fast dynamics in his objective functional. In [15,16], an attempt was made to interpret this practical multimodel situation as a perturbation problem since the "k-th model simplification" is achieved by the "k-th parameter perturbation." Under the assumptions that the fast subsystems were weakly-coupled among themselves and that each fast subsystem was affected by the control of one decision maker alone, the perturbation analysis in [15,16] established sufficient conditions for the multimodel response to be close to the actual system response. The analysis served as a basis for a decomposed design approach wherein each decision maker had to solve a separate low-order control problem in the fast time-scale, and jointly solve a low-order game problem in the slow time-scale. The two problems were solved independently to form the composite strategies which were shown to stabilize the overall system for sufficiently small values of the perturbation parameters, provided each of the low-order problems had a stabilizing solution. Furthermore, the multimodel solution was shown to be the asymptotic limit of the optimal solution, thus establishing its well-posedness.

In Chapters 2, 3, and 4, we continue to study the role of time-scales in multimodel strategy design within the framework of multiparameter singular perturbations. Specifically, we attempt to establish the validity of multimodel generation by "k-th parameter perturbation" for classes of linear deterministic systems and linear stochastic systems under nonclassical information patterns.

The structural assumptions in [15,16] correspond to practical situations where the fast subsystems are geographically distinct, each under the direct influence of one decision maker who interacts with the other decision makers only through the slow core [3]. But there might be situations where subsystem characterization by time-scales does not correspond to geographically distinct areas (in which case the fast subsystems might not be weakly-coupled); and/or a mutual relocation of controls among the decision makers might not be possible due to the inherent noncooperative nature of the problem (in which case each fast subsystem might be controlled by more than one decision maker). In Chapter 2, we examine the implications of relaxing the structural assumptions made in [15,16]. The general multiparameter game problem has been formulated in [17], and the ill-posedness of the limiting solution has been demonstrated through some examples. This happens because now the decision makers face game situations in both the fast and slow time-scales, unlike in [15,16] where they faced a control problem in the fast time-scale. In Chapter 2, we demonstrate that multimodel generation by "k-th parameter perturbation" is still well-posed provided each decision maker solves his problem by the hierarchical reduction scheme of single parameter games [21]. Unlike the multimodel solution of [15,16], the above procedure

does not guarantee stability of the overall system unless the coupling between the fast subsystems is "limited" though not necessarily "weak."

In [15,16] only deterministic problems with full state information for each decision maker were treated. The analysis involved examining the limiting solution of Riccati equations or coupled Riccati equations only. At that stage it was not quite clear whether multimodel generation by "k-th parameter perturbation" would be well-posed for stochastic problems with nonclassical information patterns where the optimal solution may involve integro-differential equations of no particular standard type. In Chapters 3 and 4 we establish the validity of such multimodel generation for a class of stochastic Nash and team problems. The weak-coupling assumption on the fast subsystems is retained to focus on aspects of randomness and nonclassical information patterns.

In Chapter 5 we consider the average-cost-per-stage problem for finite-state Markov chains with multiple decision makers. The existing results on Markov games are few [65], and do not provide us with a proper framework to study the multimodeling problem directly. For this reason we first obtain fundamental existence results for Nash and Stackelberg solutions for cases when each decision maker knows only the current value of the state, and when the leader also has access to the followers' controls at every stage. An algorithm is obtained for computing affine incentive strategy for the leader which helps him achieve his global optimum. The practical usefulness of Markovian decision processes has been severely limited due to the extremely large dimension of most Markov chains. Recent applications in queueing theory [46,47] and management of hydrodams [41,42] have exhibited Markov chain models with a "weakly-coupled" structure suitable for

perturbational analysis. In Chapter 5, after obtaining the general results, we consider a class of controlled Markov models consisting of N weakly-coupled groups of strongly-interacting states. Each group is under the authority of a single decision maker having his own performance objective and the overall system is coordinated by a leader whose objective is to optimize some global system performance. The problem considered is one where these N decision makers are in Nash equilibrium among themselves and in Stackelberg equilibrium with the leader. For the incentive design problem, it is shown that near-optimal policies can be obtained from multiple reduced-order models.

The basic challenge in multimodeling is to identify the "core" where there is a strong interaction among all the decision makers and other low-order subproblems where the interactions are weak. This leads to the possibility of decentralized strategy design by the decision makers using several low-order models of the system. Such a decomposition need not be based on time-scale considerations alone. In large scale systems, the decision makers observe, in general, different variables through their individual objective functionals. These observed variables play a crucial role in the solution of the problem. In Chapter 6, we focus on the role of the observed variables in multimodel strategy design. We attempt to identify the core by examining the observability structure induced by the observation sets of the decision makers. The system is represented in the observability decomposition form using the techniques of chained aggregation [8,54,55]. By overlapping appropriately the input structure with the observability decomposition, we identify a class of admissible strategies, referred to as Structure-Preserving strategies, which generates multimodel solutions. The information induced multimodel solutions developed in Chapter 6 are shown

to admit partial noninteraction among the decision makers under certain conditions which depend on the information pattern. Applications to the control of large scale interconnected subsystems and multi-area power systems are also discussed.

The thesis concludes with Chapter 7 where we summarize the results obtained, outline the main contributions, and indicate directions for future research.

CHAPTER 2

MULTIMODEL NASH STRATEGIES FOR MULTIPARAMETER SINGULARLY PERTURBED SYSTEMS

2.1. Introduction

Multimodel strategies for linear deterministic multiparameter singularly perturbed systems have been obtained in [15,16] under the assumption that the fast subsystems were weakly-coupled among themselves, and that each fast subsystem was affected by the control of one decision maker only. In this chapter we shall consider the general multiparameter game problem wherein the fast subsystems need not be weakly-coupled and each fast subsystem might be controlled by more than one decision maker. This problem has been formulated in [17], and the ill-posedness of the limiting solution has been demonstrated through some examples. This happens because now the decision makers face game situations in both the fast and slow time-scales, unlike in [15,16] where they faced a control problem in the fast time-scale. In the sequel we shall demonstrate that multimodel generation by "k-th parameter perturbation" is still well-posed provided each decision maker solves his problem by the hierarchical reduction scheme of single parameter games [21].

In Section 2.2 the problem is formulated and the exact solution is given. In Section 2.3 a procedure is outlined to obtain decentralized strategies from multimodel solutions. In Section 2.4, well-posedness of the multimodel solution is established; and finally in Section 2.5, the important conclusions drawn from the results of this chapter are summarized.

2.2. Problem Formulation

Consider the following linear system controlled by two decision makers

$$\dot{x} = A_{oo}x + \sum_{i=1}^{2} A_{oi}z_i + \sum_{i=1}^{2} B_{oi}u_i; \qquad x(0) = x_o \tag{2.1a}$$

$$\varepsilon_i\dot{z}_i = A_{io}x + A_{ii}z_i + A_{ij}z_j + B_{ii}u_i + B_{ij}u_j; \qquad z_i(0) = z_{io} \tag{2.1b}$$

$$i,j = 1,2; \quad i \neq j$$

dim $x = n_o$, dim $z_i = n_i$, dim $u_i = m_i$, $i = 1,2$. The small singular perturbation parameters represent small time-constants, inertias, masses etc. We consider the case when

$$m \leq \frac{\varepsilon_1}{\varepsilon_2} \leq M \tag{2.2}$$

for some positive constants m and M. Thus the set H to which we restrict the possible values of ε is a sector in R^2. The matrices A_{ii} are assumed to be nonsingular. The cost functionals of the two decision makers are

$$J_i = \frac{1}{2}\int_0^{\infty} (x'Q_{oi}x + z_i'Q_{ii}z_i + u_i'R_{ii}u_i + u_j'R_{ij}u_j)dt; \quad i,j = 1,2; \quad i \neq j. \tag{2.3}$$

The usual definiteness assumptions are made on Q_{oi}, Q_{ii}, R_{ii}, and R_{ij}. Notice that the i-th decision maker (DMi) penalizes only z_i in his cost functional, but not z_j. This is because his simplified model would neglect z_j under the multimodel situation. The decision makers select (u_1^*, u_2^*) such that

$$J_i(u_i^*, u_j^*) \leq J_i(u_i, u_j^*) \quad \text{for all admissible } u_i; \quad i,j = 1,2; \quad i \neq j. \tag{2.4}$$

The inequalities (2.4) define the Nash equilibrium.

The system model (2.1) is of interest in several cases. There might be situations where subsystem characterization by time-scales does not correspond to geographically distinct areas (in which case the fast subsystems might not be weakly-coupled); and/or a mutual relocation of controls among decision makers might not be possible due to the inherent non-cooperative nature of the problem.

The ill-posed nature of the usual order reduction method for the problem (2.1)-(2.4) was demonstrated in [17] through some examples. This is to be expected from past results on single parameter games [21-24], since now the decision makers face game situations in both the fast and slow time-scales, unlike in [15,16], when they had to solve only a control problem at the fast subsystem level. This apparently minor modification in the situation destroys the complete decoupling between the two low-order problems, and forces one to look for noncausal reduced-order models which would yield well-posed solutions.

The definitions of the various matrices that appear in the following analysis are given in Appendix A. Restricting the control strategies to be linear functions of the state, the optimal solution to (2.1)-(2.4) is given by [11]

$$u_i^* = -R_{ii}^{-1}B_i'K_i\hat{x}; \qquad \hat{x} = [x' \ \ z_1' \ \ z_2']' \tag{2.5}$$

where K_i is a stabilizing solution of the coupled Riccati equations,

$$Q_i + K_iA + A'K_i - K_iS_iK_i - K_iS_jK_j - K_jS_jK_i + K_jS_{ij}K_j = 0$$

$$i,j = 1,2; \quad i \neq j. \tag{2.6}$$

Notice that since A and B_i are functions of ε_i, K_i is also a function of ε_i. In general even for low-order problems the presence of ε causes numerical "stiffness" in (2.6). The optimal cost of each player is given by

$$J_i^* = \frac{1}{2}\hat{x}(0)'K_i(\varepsilon)\hat{x}(0); \qquad i = 1,2. \tag{2.7}$$

2.3. Multimodel Strategy Design

The notation $(\cdot)^{(i)}$ in the following formulation refers to the quantities associated with DMi's simplified problem. DMi arrives at his simplified model by neglecting the jth fast subsystem, i.e., by setting $\varepsilon_j = 0$ in (2.1). This gives

$$\bar{z}_j^{(i)} = -A_{jj}^{-1}(A_{j0}x^{(i)} + A_{ji}z_i^{(i)} + B_{ji}u_i^{(i)} + B_{jj}u_j^{(i)}). \tag{2.8}$$

Substituting (2.8) in (2.1) for z_j results in DMi's simplified model

$$\dot{x}^{(i)} = A_{oo}^{(i)}x^{(i)} + A_{oi}^{(i)}z_i^{(i)} + B_{oi}^{(i)}u_i^{(i)} + B_{oj}^{(i)}u_j^{(i)}; \qquad x^{(i)}(0) = x_o \tag{2.9a}$$

$$\varepsilon_i\dot{z}_i^{(i)} = A_{io}^{(i)}x^{(i)} + A_{ii}^{(i)}z_i^{(i)} + B_{ii}^{(i)}u_i^{(i)} + B_{ij}^{(i)}u_j^{(i)}; \qquad z_i^{(i)}(0) = z_{io}. \tag{2.9b}$$

The cost functionals of the two DMs as viewed by DMi are obtained by substituting (2.8) in (2.3)

$$J_i^{(i)} = \frac{1}{2}\int_0^\infty (x^{(i)'}Q_{oi}x^{(i)} + z_i^{(i)'}Q_{ii}z_i^{(i)} + u_i^{(i)'}R_{ii}u_i^{(i)} + u_j^{(i)'}R_{ij}u_j^{(i)})dt$$

$$\begin{aligned} J_j^{(i)} = \frac{1}{2}\int_0^\infty (&x^{(i)'}Q_{oj}^{(i)}x^{(i)} + z_i^{(i)'}Q_{jj}^{(i)}z_i^{(i)} + 2x^{(i)'}Q_{ij}^{(i)}z_i^{(i)} + 2x^{(i)'}S_j^{(i)}u_i^{(i)} \\ &+ 2x^{(i)'}P_j^{(i)}u_j^{(i)} + 2z_i^{(i)'}T_i^{(i)}u_i^{(i)} + 2z_i^{(i)'}T_j^{(i)}u_j^{(i)} + u_j^{(i)'}R_{jj}^{(i)}u_j^{(i)} \\ &+ u_i^{(i)'}R_{ji}^{(i)}u_i^{(i)} + 2u_j^{(i)'}P_{jj}^{(i)}u_i^{(i)})dt. \end{aligned} \tag{2.10}$$

We propose to solve the game (2.9)-(2.10) by the hierarchical reduction scheme of [21] which transfers fast game information to a modified slow game.

The fast subsystem is derived by assuming that the slow variables are constant during the fast transients,

$$\varepsilon_i \dot{z}_{if}^{(i)} = A_{ii}^{(i)} z_{if}^{(i)} + B_{ii}^{(i)} u_{if}^{(i)} + B_{ij}^{(i)} u_{jf}^{(i)}; \quad z_{if}^{(i)}(0) = z_{io} - \bar{z}_i^{(i)}(0). \tag{2.11}$$

The associated cost functionals are

$$J_{if}^{(i)} = \frac{1}{2}\int_0^\infty (z_{if}^{(i)'} Q_{ii} z_{if}^{(i)} + u_{if}^{(i)'} R_{ii} u_{if}^{(i)} + u_{jf}^{(i)'} R_{ij} u_{jf}^{(i)})dt$$

$$\begin{aligned} J_{jf}^{(i)} + \frac{1}{2}\int_0^\infty (z_{if}^{(i)'} Q_{jj}^{(i)} z_{if}^{(i)} &+ 2z_{if}^{(i)'} T_i^{(i)} u_{if}^{(i)} + 2z_{if}^{(i)'} T_j^{(i)} u_{jf}^{(i)} \\ &+ u_{jf}^{(i)'} R_{jj}^{(i)} u_{jf}^{(i)} + u_{if}^{(i)'} R_{ji}^{(i)} u_{if}^{(i)} + 2u_{jf}^{(i)'} P_{jj}^{(i)} u_{if}^{(i)})dt \end{aligned} \tag{2.12}$$

where $z_{if}^{(i)} = z_i^{(i)} - \bar{z}_i^{(i)}$ and $\bar{z}_i^{(i)}$ is found from (2.18).

The linear closed-loop Nash strategies for (2.11)-(2.12) are given by

$$u_{if}^{(i)} = -R_{ii}^{-1} B_{ii}^{(i)'} K_{if}^{(i)} z_{if}^{(i)} = -M_{if}^{(i)} z_{if}^{(i)} \tag{2.13a}$$

$$\begin{aligned} u_{jf}^{(i)} &= -R_{jj}^{(i)-1} [T_j^{(i)'} + B_{ij}^{(i)'} K_{jf}^{(i)} - P_{jj}^{(i)} R_{ii}^{-1} B_{ii}^{(i)'} K_{if}^{(i)}] z_{if}^{(i)} \\ &= -M_{jf}^{(i)} z_{if}^{(i)} \end{aligned} \tag{2.13b}$$

where $K_{if}^{(i)}$ and $K_{jf}^{(i)}$ are stabilizing solutions of

$$\begin{aligned} Q_{ii} + K_{if}^{(i)} A_{ii}^{(i)} + A_{ii}^{(i)'} K_{if}^{(i)} - K_{if}^{(i)} B_{ij}^{(i)} M_{jf}^{(i)} - M_{jf}^{(i)'} B_{ij}^{(i)'} K_{if}^{(i)} + M_{jf}^{(i)'} R_{ij} M_{jf}^{(i)} \\ - M_{if}^{(i)'} R_{ii} M_{if}^{(i)} = 0 \end{aligned} \tag{2.14a}$$

$$Q_{jj}^{(i)} + K_{jf}^{(i)}A_{ii}^{(i)} + A_{ii}^{(i)'}K_{jf}^{(i)} - [K_{jf}^{(i)}B_{ii}^{(i)} + T_i^{(i)}]M_{if}^{(i)} - M_{if}^{(i)'}[B_{ii}^{(i)'}K_{jf}^{(i)} + T_i^{(i)'}]$$

$$+ M_{if}^{(i)'}R_{ji}^{(i)}M_{if}^{(i)} - M_{jf}^{(i)'}R_{jj}^{(i)}M_{jf}^{(i)} = 0. \tag{2.14b}$$

Next we make use of the fast controls (2.13) and substitute the following for $u_i^{(i)}$ and $u_j^{(i)}$ in (2.9) and (2.10),

$$u_i^{(i)} = -M_{if}^{(i)}z_i^{(i)} + \hat{u}_i^{(i)} \tag{2.15a}$$

$$u_j^{(i)} = -M_{jf}^{(i)}z_i^{(i)} + \hat{u}_j^{(i)}. \tag{2.15b}$$

The new system and cost functionals are given by

$$\dot{x}^{(i)} = A_{oo}^{(i)}x^{(i)} + \hat{A}_{oi}^{(i)}z_i^{(i)} + B_{oi}^{(i)}\hat{u}_i^{(i)} + B_{oj}^{(i)}\hat{u}_j^{(i)}; \quad x^{(i)}(0) = x_o \tag{2.16a}$$

$$\varepsilon_i\dot{z}_i^{(i)} = A_{io}^{(i)}x^{(i)} + \hat{A}_{ii}^{(i)}z_i^{(i)} + B_{ii}^{(i)}\hat{u}_i^{(i)} + B_{ij}^{(i)}\hat{u}_j^{(i)}; \quad z_i^{(i)}(0) = z_{io} \tag{2.16b}$$

$$J_i^{(i)} = \frac{1}{2}\int_0^\infty \{x^{(i)'}Q_{oi}x^{(i)} + z_i^{(i)'}\hat{Q}_{ii}z_i^{(i)} - 2z_i^{(i)'}(M_{if}^{(i)'}R_{ii}\hat{u}_i^{(i)} + M_{jf}^{(i)'}R_{ij}\hat{u}_j^{(i)})$$

$$+ \hat{u}_i^{(i)'}R_{ii}\hat{u}_i^{(i)} + \hat{u}_j^{(i)'}R_{ij}\hat{u}_j^{(i)}\}dt \tag{2.17a}$$

$$J_j^{(i)} = \frac{1}{2}\int_0^\infty \{x^{(i)'}Q_{0j}^{(i)}x^{(i)} + 2x^{(i)'}\hat{Q}_{ij}^{(i)}z_i^{(i)} + z_i^{(i)'}\hat{Q}_{jj}^{(i)}z_i^{(i)} + 2x^{(i)'}S_j^{(i)}\hat{u}_i^{(i)}$$

$$+ 2x^{(i)'}P_j^{(i)}\hat{u}_j^{(i)} + 2z_i^{(i)'}\hat{T}_i^{(i)}\hat{u}_i^{(i)} + 2z_i^{(i)'}\hat{T}_j^{(i)}\hat{u}_j^{(i)}$$

$$+ \hat{u}_j^{(i)'}R_{jj}^{(i)}\hat{u}_j^{(i)} + \hat{u}_i^{(i)'}R_{ji}^{(i)}\hat{u}_i^{(i)} + 2\hat{u}_j^{(i)'}P_{jj}^{(i)}\hat{u}_i^{(i)}\}dt. \tag{2.17b}$$

Now we set $\varepsilon_i = 0$ in (2.16b) and solve for $\bar{z}_i^{(i)}$,

$$\bar{z}_i^{(i)} = -\hat{A}_{ii}^{(i)^{-1}}(A_{io}^{(i)}x_s^{(i)} + B_{ii}^{(i)}\hat{u}_{is}^{(i)} + B_{ij}^{(i)}\hat{u}_{js}^{(i)}). \tag{2.18}$$

Substituting (2.18) for $\bar{z}_i^{(i)}$ in (2.16a) and (2.17), the slow subsystem and cost functionals are obtained as

$$\dot{x}_s^{(i)} = A_{os}^{(i)} x_s^{(i)} + B_{is}^{(i)} \hat{u}_{is}^{(i)} + B_{js}^{(i)} \hat{u}_{js}^{(i)}; \qquad x_s^{(i)}(0) = x_o \tag{2.19}$$

$$J_{is}^{(i)} = \frac{1}{2} \int_0^\infty \{ x_s^{(i)'} Q_{is}^{(i)} x_s^{(i)} + 2 z_s^{(i)'} S_{is}^{(i)} \hat{u}_{is}^{(i)} + 2 x_s^{(i)'} P_{is}^{(i)} \hat{u}_{js}^{(i)}$$

$$+ \hat{u}_{is}^{(i)'} R_{ii} \hat{u}_{is}^{(i)} + \hat{u}_{js}^{(i)'} R_{ijs} \hat{u}_{js}^{(i)} + 2 \hat{u}_{is}^{(i)'} P_{iis}^{(i)} \hat{u}_{js}^{(i)} \} dt \tag{2.20a}$$

$$J_{js}^{(i)} = \frac{1}{2} \int_0^\infty \{ x_s^{(i)'} Q_{js}^{(i)} x_s^{(i)} + 2 x_s^{(i)'} S_{js}^{(i)} \hat{u}_{is}^{(i)} + 2 x_s^{(i)'} P_{js}^{(i)} \hat{u}_{js}^{(i)} + \hat{u}_{js}^{(i)'} R_{jj}^{(i)} \hat{u}_{js}^{(i)}$$

$$+ \hat{u}_{is}^{(i)'} R_{jis}^{(i)} \hat{u}_{is}^{(i)} + 2 \hat{u}_{js}^{(i)'} P_{jjs}^{(i)} \hat{u}_{is}^{(i)} \} dt. \tag{2.20b}$$

Notice that the slow subsystem and the associated cost functionals contain information about the fast game. The linear closed-loop Nash strategy for (2.19)-(2.20) is given by

$$\hat{u}_{is}^{(i)} = -R_{ii}^{-1} [S_{is}^{(i)} x_s^{(i)} + B_{is}^{(i)'} K_{is}^{(i)} x_s^{(i)} + P_{iis}^{(i)} \hat{u}_{js}^{(i)}] = -M_{is}^{(i)} x_s^{(i)} \tag{2.21a}$$

$$\hat{u}_{js}^{(i)} = -R_{jj}^{(i)-1} [P_{js}^{(i)} x_s^{(i)} + B_{js}^{(i)'} K_{js}^{(i)} x_s^{(i)} + P_{jjs}^{(i)} \hat{u}_{is}^{(i)}] = -M_{js}^{(i)} x_s^{(i)} \tag{2.21b}$$

where $K_{is}^{(i)}$ and $K_{js}^{(i)}$ are stabilizing solutions of

$$Q_{is}^{(i)} + K_{is}^{(i)} A_{os}^{(i)} + A_{os}^{(i)'} K_{is}^{(i)} - [K_{is}^{(i)} B_{js}^{(i)} + P_{is}^{(i)}] M_{js}^{(i)} - M_{js}^{(i)'} [K_{is}^{(i)} B_{js}^{(i)} + P_{is}^{(i)}]'$$

$$+ M_{js}^{(i)'} R_{ijs} M_{js}^{(i)} - M_{is}^{(i)'} R_{ii} M_{is}^{(i)} = 0 \tag{2.22a}$$

$$Q_{js}^{(i)} + K_{js}^{(i)} A_{os}^{(i)} + A_{os}^{(i)'} K_{js}^{(i)} - [K_{js}^{(i)} B_{is}^{(i)} + S_{js}^{(i)}] M_{is}^{(i)} - M_{is}^{(i)'} [K_{js}^{(i)} B_{is}^{(i)} + S_{js}^{(i)}]'$$

$$+ M_{is}^{(i)'} R_{jis}^{(i)} M_{is}^{(i)} - M_{js}^{(i)'} R_{jj}^{(i)} M_{js}^{(i)} = 0. \tag{2.22b}$$

Hence, the composite strategies for the simplified game of DMi are given by

$$u_i^{(i)} = -M_{is}^{(i)} x^{(i)} - M_{if}^{(i)} z_i^{(i)}$$

$$u_j^{(i)} = -M_{js}^{(i)} x^{(i)} - M_{jf}^{(i)} z_i^{(i)}; \qquad i,j = 1,2;\ i \neq j. \tag{2.23}$$

The decentralized multimodel strategy which the two decision makers use on the full system (2.1), as obtained from the two simplified games, is then given by

$$u_i^c = -M_{is}^{(i)} x - M_{if}^{(i)} z_i \quad ; \qquad i = 1,2. \tag{2.24}$$

Remarks:

1) The slow and fast subproblems are both game problems, and are different for both the players. This is in contrast to the weakly-coupled problem considered in [15,16] where only the fast subproblems, which were control problems, were different for the two players; whereas the slow subproblem, which was a game problem, was the same for both the players.

2) The system and cost matrices of the slow subproblems of both players contain information about their respective fast games, highlighting the "anticipative" nature of low-order models in multiple decision maker problems. This is again in contrast to the weakly-coupled case of [15,16] where the fast and slow subproblems were solved independently.

3) The multimodel solution of [15,16] did guarantee the stability of the overall system for all ε in H, but the multimodel strategy (2.24) obtained when the fast subsystems are not weakly-coupled does not guarantee stability, unless the coupling is limited (not necessarily weak). Therefore, the following assumption is made:

<u>Assumption A</u>:

The solutions to the reduced games exist, and when the multimodel strategy pair (u_1^c, u_2^c) is applied to the original system (2.1), the closed-loop system remains asymptotically stable for all ε in H.

2.4. Asymptotic Properties of the Multimodel Strategy

In this section, we shall show that the multimodel strategy and the resulting costs are well-posed in the sense that they tend to the optimal strategy and costs respectively in the limit as the small parameters ε_i go to zero.

The multimodel strategy (2.24) is put in a convenient form as follows:

$$u_1^c = -R_{11}^{-1}[B'_{01}\;\; B'_{11}/\varepsilon_1\;\; B'_{21}/\varepsilon_2]\begin{bmatrix} K_{1s}^{(1)} & 0 & 0 \\ \varepsilon_1 K'_{1m} & \varepsilon_1 K_{1f}^{(1)} & 0 \\ -\varepsilon_2[(A_{02}A_{22}^{-1})'K_{1s}^{(1)} + (A_{12}A_{22}^{-1})'K'_{1m}] & -\varepsilon_2(A_{12}A_{22}^{-1})'K_{1f}^{(1)} & 0 \end{bmatrix}\begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix}$$

$$= -R_{11}^{-1}B'_1 L_1 \hat{x} \tag{2.25a}$$

$$u_2^c = -R_{22}^{-1}[B'_{02}\;\; B'_{12}/\varepsilon_1\;\; B'_{22}/\varepsilon_2]\begin{bmatrix} K_{2s}^{(2)} & 0 & 0 \\ -\varepsilon_1[(A_{01}A_{11}^{-1})'K_{2s}^{(2)} + (A_{21}A_{11}^{-1})'K'_{2m}] & 0 & -\varepsilon_1(A_{21}A_{11}^{-1})'K_{2f}^{(2)} \\ \varepsilon_2 K'_{2m} & 0 & \varepsilon_2 K_{2f}^{(2)} \end{bmatrix}\begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix}$$

$$= -R_{22}^{-1}B'_2 L_2 \hat{x} \tag{2.25b}$$

where

$$K_{im} = [-A_{io}^{(i)'}K_{if}^{(i)}-K_{is}^{(i)}A_{oi}^{(i)} + \{K_{js}^{(i)}B_{js}^{(i)}+P_{js}^{(i)}-(K_{is}^{(i)}B_{is}^{(i)}+S_{is}^{(i)})R_{ii}^{-1}P_{jjs}^{(i)}\}\cdot$$

$$\{R_{jj}^{(i)}-P_{iis}^{(i)}R_{ii}^{-1}P_{jjs}^{(i)'}\}^{-1}\{B_{ij}^{(i)}K_{if}^{(i)}+R_{ij}M_{if}^{(i)}\}]A_{os}^{(i)^{-1}}$$

$$i,j = 1,2; \quad i \neq j. \tag{2.26}$$

To avoid unboundedness in the solution of (2.6) as $\varepsilon \to 0$ in H, and taking into consideration the symmetry of K_1, K_2 and the special forms of A, B_1, B_2, we seek the solutions K_i of (2.6) in the form

$$K_i(\varepsilon) = \begin{bmatrix} K_{00}^{(i)}(\varepsilon) & \varepsilon_1 K_{01}^{(i)}(\varepsilon) & \varepsilon_2 K_{02}^{(i)}(\varepsilon) \\ \varepsilon_1 K_{01}^{(i)'}(\varepsilon) & \varepsilon_1 K_{11}^{(i)}(\varepsilon) & \sqrt{\varepsilon_1\varepsilon_2}\, K_{12}^{(i)}(\varepsilon) \\ \varepsilon_2 K_{02}^{(i)'}(\varepsilon) & \sqrt{\varepsilon_1\varepsilon_2}\, K_{12}^{(i)'}(\varepsilon) & \varepsilon_2 K_{22}^{(i)}(\varepsilon) \end{bmatrix}; \; i=1,2. \tag{2.27}$$

<u>Theorem 2.1</u>: The following relations hold under Assumption A:

$$K_{oo}^{(i)}(0) = K_{is}^{(i)}$$

$$K_{oi}^{(i)}(0) = K_{im}$$

$$K_{ii}^{(i)}(0) = K_{if}^{(i)}$$

$$K_{jj}^{(i)}(0) = 0$$

$$K_{oj}^{(i)}(0) = -[(A_{oj}A_{jj}^{-1})'K_{is}^{(i)} + (A_{ij}A_{jj}^{-1})'K_{im}']'; \qquad i,j=1,2; \; i \neq j$$

$$K_{12}^{(1)}(0) = -\sqrt{\alpha_{21}}\, K_{1f}^{(1)}(A_{12}\, A_{22}^{-1})$$

$$K_{12}^{(2)}(0) = -\sqrt{\alpha_{12}}\, (A_{21}\, A_{11}^{-1})'K_{2f}^{(2)}$$

where

$$\alpha_{ij} = \lim_{\|\varepsilon\|\to 0} \left(\frac{\varepsilon_i}{\varepsilon_j}\right).$$

Proof: The proof involves substituting (2.27) in (2.6) and taking the limit as $\|\varepsilon\| \to 0$. The detailed manipulations are lengthy and are omitted here for the sake of brevity.

Corollary 2.1: If the multimodel strategies exist, then

$$\lim_{\|\varepsilon\|\to 0} (L_i(\varepsilon)-K_i(\varepsilon)) = 0.$$

Proof: The result is an immediate consequence of Theorem 2.1.

When the multimodel strategy (u_1^c, u_2^c) is applied to (2.1), the resulting cost is given by

$$J_i^c = \frac{1}{2}\,\hat{x}(0)'V_i(\varepsilon)\hat{x}(0); \quad i = 1,2 \quad ; \tag{2.28}$$

where $V_i(\varepsilon)$ satisfies the Lyapunov equation

$$V_i(A-S_1L_1-S_2L_2) + (A-S_1L_1-S_2L_2)'V_i + Q_i + L_i'S_iL_i + L_j'S_{ij}L_j = 0. \tag{2.29}$$

By Assumption A, $V_i(\varepsilon)$ exists and is positive definite for all ε in H.

Lemma 2.1:

$$J_i^c = J_i^* + 0(\|\varepsilon\|); \quad i = 1,2, \quad \forall\varepsilon \text{ in H}.$$

Proof: Subtracting (2.6) from (2.29) and letting $W_i = V_i - K_i$, we get

$$\begin{aligned} W_i(A-S_1L_1-S_2L_2) + (A-S_1L_1-S_2L_2)'W_i + (K_i-L_i)'S_i(K_i-L_i) + (K_j-L_j)'S_{ij}(K_j-L_j) \\ + K_jS_j(K_j-L_j) + (K_j-L_j)'S_jK_j + K_jS_{ij}(L_j-K_j) + (L_j-K_j)'S_{ij}K_j = 0. \end{aligned} \tag{2.30}$$

From Corollary 2.1 and Assumption A, we get

$$\lim_{\|\varepsilon\|\to 0} W_i = 0; \qquad i = 1,2$$

and hence $$J_i^c = J_i^* + O(\|\varepsilon\|); \qquad i = 1,2 \quad \forall\varepsilon \text{ in } H.$$

We have proposed that the multimodel strategy (u_1^c, u_2^c) be used as an approximation of the exact Nash strategy (u_1^*, u_2^*). It is not clear at this point why decision makers, who are interested in a Nash strategy should use the multimodel strategy. The exact Nash strategy (u_1^*, u_2^*) satisfies inequality (2.4), which guarantees that neither decision maker can reduce his cost functional by unilaterally deviating from (u_1^*, u_2^*). Unfortunately, the multimodel strategy does not possess this property, and hence it is necessary to establish its near-equilibrium property [20]. We have shown that the resulting costs of the multimodel strategy are $O(\|\varepsilon\|)$ close to their Nash equilibrium values. However, closeness of the costs alone is not sufficient. If player-i uses u_i^c, player-j solves an optimal control problem in u_j. The strategy u_j^c must be a near-optimal strategy for this optimal control problem, otherwise player-j would have no motive for using u_j^c. This guarantees that the j-th player cannot reduce his cost by more than $O(\|\varepsilon\|)$ if he unilaterally deviates from (u_1^c, u_2^c). Hence, practically the players have no motive for cheating. This, however, is not a guarantee against cheating. It is quite possible that the j-th player deviates from u_j^c and uses another strategy $\hat{u}_j$ that reduces his cost J_j no matter how insignificant the reduction is; but in doing so hurts the other player by causing a substantial increase in J_i. Hence, for (u_1^c, u_2^c) to qualify as a near-equilibrium strategy pair, it must be true that any $\hat{u}_j$ that results in $J_j(u_i^c, \hat{u}_j) \le J_j(u_i^c, u_j^c)$ cannot increase J_i by more than $O(\|\varepsilon\|)$. The definition of a near-equilibrium strategy as given in [20] does not require the existence of a Nash equilibrium strategy. Here we shall

show that the proposed multimodel strategy (u_1^c, u_2^c) is not just near-equilibrium Nash, but being $O(\|\varepsilon\|)$ close to (u_1^*, u_2^*), is also asymptotic Nash.

Define the set of admissible strategies for player 1, when player 2 uses u_2^c, as the set of linear feedback strategies of the form,

$$u_1 = -F_1(\varepsilon)\hat{x} = -(F_{10}(\varepsilon)x + F_{11}(\varepsilon)z_1 + F_{12}(\varepsilon)z_2) \tag{2.31}$$

such that the closed-loop matrix

$$A_c = (A - B_1F_1 - S_2L_2)$$

is stable for all ε in H. To avoid mathematical complications, the feedback matrices of (2.31) are restricted to be of the form,

$$F_{1i}(\varepsilon) = \bar{F}_{1i} + O(\|\varepsilon\|); \qquad i = 0,1,2. \tag{2.32}$$

Denote this set by U_1. The set of admissible strategies for player-2 when player-1 uses u_1^c is similarly defined and is denoted by U_2.

The following lemma is needed to establish the near-equilibrium Nash property of the multimodel strategy.

<u>Lemma 2.2</u>:

$$J_1(u_1, u_2^c) - J_1(u_1, u_2^*) = O(\|\varepsilon\|); \quad \forall u_1 \in U_1, \quad \varepsilon \text{ in H}.$$

<u>Proof</u>: Let

$$J_1(u_1, u_2^*) = \frac{1}{2}\hat{x}'(0)T_1\hat{x}(0) \tag{2.33}$$

where T_1 satisfies

$$T_1(A - B_1F_1 - S_2K_2) + (A - B_1F_1 - S_2K_2)'T_1 + Q_1 + F_1'R_{11}F_1 + K_2S_{12}K_2 = 0 \tag{2.34}$$

and

$$J_1(u_1, u_2^c) = \frac{1}{2}\hat{x}(0)'P_1\hat{x}(0) \tag{2.35}$$

where P_1 satisfies

$$P_1(A-B_1F_1-S_2L_2)+(A-B_1F_1-S_2L_2)'P_1+Q_1+F_1'R_{11}F_1+L_2'S_{12}L_2=0\ . \qquad (2.36)$$

Subtracting (2.34) from (2.36) and letting $N_1=P_1-T_1$ we get

$$N_1(A-B_1F_1-S_2L_2)+(A-B_1F_1-S_2L_s)'N_1+T_1S_2(K_2-L_2)+(K_2-L_2)'S_2T_1$$

$$+(K_2-L_2)'S_{12}(K_2-L_2)+K_2S_{12}(L_2-K_2)+(L_2-K_2)'S_{12}K_2=0\ . \qquad (2.37)$$

From Corollary 2.1, and knowing the stability of $(A-B_1F_1-S_2L_2)$ $\forall\varepsilon$ in H, we get

$$\lim_{\|\varepsilon\|\to 0} N_1 = 0$$

which proves Lemma 2.2.

The following two theorems establish the near-equilibrium property of the multimodel strategy.

<u>Theorem 2.2</u>:

$$J_i(u_i^c,u_j^c)\le J_i(u_i,u_j^c)+0(\|\varepsilon\|);\ \forall u_i\in U_i,\ \ \varepsilon \text{ in } H;\ i,j=1,2,\ i\neq j$$

i.e., the multimodel strategy is almost secure against cheating.

<u>Proof</u>: We have

$$J_1(u_1^c,u_2^c) = J_1(u_1,u_2^c) + J_1(u_1^c,u_2^c) - J_1(u_1^*,u_2^*) + J_1(u_1^*,u_2^*) - J_1(u_1,u_2^c)$$

Since $J_1(u_1^*,u_2^*)\le J_1(u_1,u_2^*)$, we get

$$J_1(u_1^c,u_2^c) \le J_1(u_1,u_2^c) + J_1(u_1^c,u_2^c) - J_1(u_1^*,u_2^*) + J_1(u_1,u_2^*) - J_1(u_1,u_2^c)$$

From Lemma 2.1 and Lemma 2.2, it follows that

$$J_1(u_1^c,u_2^c) \leq J_1(u_1,u_2^c) + O(\|\varepsilon\|) \quad \forall u_1 \in U_1 \quad .$$

This proves the theorem for $i=1$, $j=2$. The other case is similar.

<u>Theorem 2.3</u>:

$$J_i(u_i^c,\hat{u}_j) \leq J_i(u_i^c,u_j^c) + O(\|\varepsilon\|);$$

$$\forall \hat{u}_j \in U_j \quad \text{such that} \quad J_j(u_i^c,\hat{u}_j) \leq J_j(u_i^c,u_j^c)$$

$$\forall \varepsilon \in H;\ i,j = 1,2;\ i \neq j$$

<u>Proof</u>: We prove for $i=2$, $j=1$. The other case is similar. Suppose player-2 uses $u_2^c = -R_{22}^{-1}B_2'L_2\hat{x}$; the optimal reaction of player-1 is given by

$$\hat{u}_1^* = -R_{11}^{-1}B_1'M_1\hat{x} \tag{2.38}$$

resulting in

$$J_1(\hat{u}_1^*,u_2^c) = \frac{1}{2}\,\hat{x}(0)'M_1\hat{x}(0) \tag{2.39}$$

where M_1 satisfies

$$M_1(A - S_1M_1 - S_2L_2) + (A - S_1M_1 - S_2L_2)'M_1 + Q_1 + M_1S_1M_1 + L_2'S_{12}L_2 = 0. \tag{2.40}$$

Subtracting (2.40) from (2.29) for $i=1$, $j=2$ and letting $\Phi = V_1 - M_1$ we have

$$\Phi(A - S_1L_1 - S_2L_2) + (A - S_1L_1 - S_2L_2)'\Phi + M_1S_1(M_1 - L_1) + L_1'S_1(L_1 - M_1) = 0. \tag{2.41}$$

It follows from Theorem 2.2 that

$$J_1(\hat{u}_1^*,u_2^c) - J_1(u_1^c,u_2^c) = O(\|\varepsilon\|) \tag{2.42}$$

or

$$\lim_{\|\varepsilon\| \to 0} \Phi(\varepsilon) = 0 \tag{2.43}$$

and hence to satisfy (2.41) we should have

$$\lim_{\|\varepsilon\|\to 0} (L_1 - M_1) = 0. \tag{2.44}$$

Let $\hat{u}_1 = -F_1\hat{x}$, $\hat{u}_1 \in U_1$ be any strategy such that

$$J_1(\hat{u}_1, u_2^c) = \frac{1}{2}\,\hat{x}(0)'D_1\hat{x}(0) \le J_1(u_1^c, u_2^c) = \frac{1}{2}\,\hat{x}(0)'V_1\hat{x}(0) \tag{2.45}$$

D_1 satisfies the Lyapunov equation

$$D_1(A - B_1F_1 - S_2L_2) + (A - B_1F_1 - S_2L_2)'D_1 + Q_1 + F_1'R_{11}F_1 + L_2'S_{12}L_2 = 0. \tag{2.46}$$

Subtracting (2.40) from (2.46), $\Psi_1 = D_1 - M_1$ satisfies

$$\Psi_1(A - B_1F_1 - S_2L_2) + (A - B_1F_1 - S_2L_2)'\Psi_1 + \delta = 0\,; \tag{2.47}$$

where

$$\delta = (R_{11}^{-1}B_1'M_1 - F_1)'R_{11}(R_{11}^{-1}B_1'M_1 - F_1)\,. \tag{2.48}$$

From (2.45) it follows that

$$0 \le \hat{x}(0)'\Psi_1(\varepsilon)\hat{x}(0) \le \hat{x}(0)'\Phi(\varepsilon)\hat{x}(0)\,. \tag{2.49}$$

Hence, due to (2.43) we get

$$\lim_{\|\varepsilon\|\to 0} \Psi_1(\varepsilon) = 0 \quad ; \tag{2.50}$$

and therefore, from (2.47) it follows that

$$\lim_{\|\varepsilon\|\to 0} (R_{11}^{-1}B_1'M_1 - F_1) = 0\,. \tag{2.51}$$

Equations (2.44) and (2.51) show that any strategy $\hat{u}_1$ satisfying (2.45) must satisfy

$$\lim_{\|\varepsilon\|\to 0} (R_{11}^{-1}B_1'L_1 - F_1) = 0. \tag{2.52}$$

Now,

$$J_2(\hat{u}_1,u_2^c) = \frac{1}{2}\,\hat{x}(0)'D_2\hat{x}(0), \quad J_2(u_1^c,u_2^c) = \frac{1}{2}\hat{x}(0)'V_2\hat{x}(0) \; ; \tag{2.53}$$

where D_2 satisfies the Lyapunov equation

$$D_2(A - B_1F_1 - S_2L_2) + (A - B_1F_1 - S_2L_2)'D_2 + Q_2 + F_1'R_{21}F_1 + L_2'S_2L_2 = 0 . \tag{2.54}$$

Subtracting (2.29) for i=2, j=1 from (2.54) and letting $\Psi_2 = D_2 - V_2$ we get

$$\Psi_2(A - B_1F_1 - S_2L_2) + (A - B_1F_1 - S_2L_2)'\Psi_2 + V_2B_1(R_{11}^{-1}B_1'L_1 - F_1) + (R_{11}^{-1}B_1'L_1 - F_1)'B_1'V_2$$
$$+ (R_{11}^{-1}B_1'L_1 - F_1)'R_{21}(R_{11}^{-1}B_1'L_1 - F_1) + L_1'B_1R_{11}^{-1}R_{21}(F_1 - R_{11}^{-1}B_1'L_1)$$
$$+ (F_1 - R_{11}^{-1}B_1'L_1)'R_{21}R_{11}^{-1}B_1'L_1 = 0 . \tag{2.55}$$

From (2.52) and knowing the stability of $(A-B_1F_1-S_2L_2)$ it follows that

$$\lim_{\|\varepsilon\|\to 0} \Psi_2 = 0 \tag{2.56}$$

which proves the theorem for $i=2$, $j=1$.

By a simple modification, the multimodel strategy (2.24) can be reformulated as a linear function of the slow state alone. To obtain DMi's modified multimodel strategy, we substitute (2.21) into (2.18) to give

$$\bar{z}_i^{(i)} = -\hat{A}_{ii}^{(i)-1}(A_{io}^{(i)} - B_{ii}^{(i)}M_{is}^{(i)} - B_{ij}^{(j)}M_{js}^{(i)})x_s^{(i)} . \tag{2.57}$$

Substituting (2.57) in (2.24) for $z_i^{(i)}$ we obtain,

$$u_{i\ell}^c = -[M_{is}^{(i)} - M_{if}^{(i)}\hat{A}_{ii}^{(i)-1}(A_{io}^{(i)} - B_{ii}^{(i)}M_{is}^{(i)} - B_{ij}^{(i)}M_{js}^{(i)})]x_s . \tag{2.58}$$

This can be factorized to put into a convenient form,

$$u_{1\ell}^{c} = -R_{11}^{-1}[B_{01}' \; B_{11}'/\varepsilon_1 \; B_{21}'/\varepsilon_2]\begin{bmatrix} K_{1s}^{(1)} & 0 & 0 \\ \varepsilon_1\bar{K}_{1m}' & 0 & 0 \\ -\varepsilon_2[(A_{02}A_{22}^{-1})'K_{1s}^{(1)} & & \\ +(A_{12}A_{22}^{-1})'\bar{K}_{1m}'] & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix},$$

$$= -R_{11}^{-1}B_1'\tilde{L}_1\hat{x} \;; \tag{2.59a}$$

$$u_{2\ell}^{c} = -R_{22}^{-1}[B_{02}' \; B_{12}'/\varepsilon_1 \; B_{22}'/\varepsilon_2]\begin{bmatrix} K_{2s}^{(2)} & 0 & 0 \\ -\varepsilon_1[(A_{01}A_{11}^{-1})'K_{2s}^{(2)} & & \\ +(A_{21}A_{11}^{-1})'\bar{K}_{2m}'] & 0 & 0 \\ \varepsilon_2\bar{K}_{2m}' & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix},$$

$$= -R_{22}^{-1}B_2'\tilde{L}_2\hat{x} \;; \tag{2.59b}$$

where

$$\bar{K}_{im} = K_{im} - (A_{io}^{(i)} - B_{ii}^{(i)}M_{is}^{(i)} - B_{ij}^{(i)}M_{js}^{(i)})'\hat{A}_{ii}^{(i)^{-1}}K_{if}^{(i)}; \quad i,j=1,2; \quad i\neq j. \tag{2.60}$$

The resulting cost, when the modified multimodel strategy is applied to (2.1), can be written as

$$J_{i\ell} = \frac{1}{2}\hat{x}(0)'V_{i\ell}\hat{x}(0); \quad i=1,2 \;; \tag{2.61}$$

where $V_{i\ell}$ satisfies the Lyapunov equation

$$V_{i\ell}(A - S_1\tilde{L}_1 - S_2\tilde{L}_2) + (A - S_1\tilde{L}_1 - S_2\tilde{L}_2)'V_{i\ell} + Q_i + \tilde{L}_i'S_i\tilde{L}_i + \tilde{L}_j'S_{ij}\tilde{L}_j = 0 \;; \quad i,j=1,2;\; i\neq j \tag{2.62}$$

If $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is block D-stable [18], then from Assumption A it follows that

$(A - S_1\tilde{L}_1 - S_2\tilde{L}_2)$ is stable for all ε in H; and hence $V_{i\ell}$ exists and is positive definite for all ε in H. Following the methods used earlier, it can be shown that

$$J_{i\ell} = J_i^* + O(\|\varepsilon\|); \quad i = 1,2 \quad \forall\varepsilon \text{ in H.} \tag{2.63}$$

The modified multimodel strategy also possesses the near-equilibrium property. This is true because

$$J_i(u_i, u_{j\ell}^c) - J_i(u_i, u_j^*) = O(\|\varepsilon\|) \quad \forall u_i \in U_i, \quad \varepsilon \text{ in H}; \ i,j = 1,2; \ i \neq j. \tag{2.64}$$

The above fact follows directly from the discussions in [25], and Lemma 2.2.

Hence, together with (2.63) and (2.64) we establish the near-equilibrium property of the modified multimodel strategy, namely,

$$J_i(u_{i\ell}^c, u_{j\ell}^c) \leq J_i(u_i, u_{j\ell}^c) + O(\|\varepsilon\|); \ \forall u_i \in U_i, \ \varepsilon \text{ in H}; \ i,j = 1,2; \ i \neq j. \tag{2.65}$$

$$J_i(u_{i\ell}^c, \hat{u}_j) \leq J_i(u_{i\ell}^c, u_{j\ell}^c) + O(\|\varepsilon\|); \ \forall \hat{u}_j \in U_j \text{ such that } J_j(u_{i\ell}^c, \hat{u}_j) \leq J_j(u_{i\ell}^c, u_{j\ell}^c)$$
$$\forall\varepsilon \text{ in H}; \ i,j = 1,2; \ i \neq j \tag{2.66}$$

Finally, we would like to remark that though the approximate strategies derived in this paper possess near-equilibrium and asymptotic Nash properties, the resulting state trajectories are within $O(\|\varepsilon\|)$ of the optimal trajectories only outside some boundary-layer.

2.5. Conclusions

In this chapter a procedure has been formulated to obtain decentralized strategies under a multimodel situation. The proposed strategies are near-equilibrium and asymptotic Nash. The subsystem classification was based on time-scale separation, which allowed the system to be modeled with multiparameter singular perturbations. The weak-coupling assumption made on the fast subsystems in [15,16] was removed. This apparently minor modification in the model structure changed completely the multimodel solution procedure. The reduced games for the two players became completely different, in contrast to the problem in [15,16] where the two reduced games were only partially different, the difference being in the fast control problems; the slow game problems being identical for both the players. Moreover, the multimodel solution in [15,16] guaranteed the stability of the overall system for all ε in H; but the multimodel solution proposed here,under the absence of weak-coupling, failed to guarantee the stability of the overall system unless the coupling between the fast subsystems is limited (not necessarily weak). In the case when the boundary-layer system is asymptotically stable for all ε in H (block D-stable), a procedure is given to modify the multimodel strategies to obtain strategies which are linear functions of the slow state alone. These modified strategies are also near-equilibrium and asymptotic Nash.

CHAPTER 3

A MULTIMODEL APPROACH TO STOCHASTIC NASH GAMES

3.1. Introduction

In this chapter we establish the well-posedness of multimodel generation by "k-th parameter perturbation" for a class of stochastic Nash games with a prespecified finite-dimensional compensator structure for each decision maker. The weak-coupling assumption is retained to keep the analysis simple, and focus on the stochastic aspects of the problem.

In Section 3.2 we formulate the problem and raise some crucial questions. Section 3.3 demonstrates multimodel generation. In Section 3.4 we establish the weak limit of the fast stochastic variable. In Section 3.5 we solve the slow subproblem and in Section 3.6 we solve the fast subproblems. In Section 3.7 we examine the limiting behavior of the exact solution and establish the well-posedness of the multimodel solution. Finally, in Section 3.8, we conclude the chapter by summarizing the main results.

3.2. Problem Formulation

A linear stochastic system consisting of a strongly-coupled slow core and weakly-coupled fast subsystems controlled by two decision makers is modeled by

$$\dot{z}_o = A_o z_o + \sum_{j=1}^{2} A_{oj} z_j + \sum_{j=1}^{2} B_{oj} u_j + L_o w; \qquad z_o(0) = z_{oo}. \tag{3.1a}$$

$$\varepsilon_i \dot{z}_i = A_{io} z_o + A_{ii} z_i + \varepsilon_{ij} A_{ij} z_j + B_{ii} u_i + \sqrt{\varepsilon_i} L_i w; \qquad z_i(0) = z_{io};$$

$$i,j = 1,2; \quad i \neq j . \tag{3.1b}$$

with the observation vectors for each decision maker given by

$$y_{oi} = C_{0i}z_o + v_{oi} \tag{3.2a}$$

$$y_{ii} = C_{ii}z_i + \sqrt{\varepsilon_i}v_{ii} \quad ; \qquad i=1,2 . \tag{3.2b}$$

where $\dim z_o = n_o$, $\dim z_i = n_i$, $\dim u_i = m_i$, $\dim y_{oi} = P_{oi}$, $\dim y_{ii} = P_{ii}$; $i=1,2$. The processes w, v_{oi}, v_{ii} are assumed to be independent white Gaussian with covariances W, V_{oi}, and V_{ii} respectively, with positive definite V_{oi} and V_{ii}. The initial conditions are assumed to be Gaussian with

$$E[z_{io}] = \bar{z}_{io} \quad ,$$

$$E[(z_{io}-\bar{z}_{io})(z_{jo}-\bar{z}_{jo})'] = N_{ij} \quad ; \qquad i,j = 0,1,2. \tag{3.3}$$

The small singular perturbation parameters ε_i represent small time-constants, inertias, masses etc.; while the small regular perturbation parameters ε_{ii} represent weak-coupling between the fast subsystems. The states z_i are "fast" since their derivatives are of order $1/\varepsilon_i$. The matrices A_{ii} are nonsingular.

The main idea behind inserting the $\sqrt{\varepsilon_i}$ factor multiplying the white noise terms in the state and observation equations for the variables z_1, z_2 is to make them meaningful fast variables for control and estimation purposes. Without $\sqrt{\varepsilon_i}$ in the state equation, the variable $z_i(t)$ tends to a white noise vector with infinite variance parameter as $\varepsilon_i \to 0$. If this factor is dropped from the observations equation, then $z_i(t)$ cannot be estimated meaningfully because the signal-to-noise ratio tends to zero as $\varepsilon_i \to 0$. A more complete discussion about the use and justification of this model can be found in [33].

The cost functionals of the two decision makers are given by

$$J_i = \frac{1}{2}\{z_o'(T)\bar{\Gamma}_{oi}z_o(T) + \varepsilon_i z_i'(T)\bar{\Gamma}_i z_i(T) + \int_o^T (z_o'\bar{Q}_{oi}z_o + z_i'\bar{Q}_i z_i + u_i'R_i u_i)dt\} \quad ; \quad i = 1,2. \tag{3.4}$$

The equilibrium solution to the stochastic zero-sum game under general information structures has been obtained in [26,29]. The solution has been shown to require infinite-dimensional compensators which are not practical to implement. Although the general solution to the nonzero-sum Nash game has not yet appeared in the literature, it appears however, that infinite-dimensional compensators would still be required. In such a case, one can either make specific assumptions regarding the information structures of the two players, under which the required compensators turn out to be finite-dimensional dynamic systems [28]; or solve the problem under the formal restriction that each player is limited to a compensator of fixed dimension, the output of which is all that is available to him in the generation of his control at that time [35].

Our intention here is not to solve the general LQG Nash game, but to obtain approximate limiting strategies for a given solution methodology. For this purpose, we extend the results of [35] for the constrained estimator problem, to two-person nonzero-sum LQG Nash games and based on this solution methodology obtain the limiting strategies under a multimodel situation. Our motivation in taking the above approach is that finite-dimensional estimators are practical to implement, and possess some nice properties.

The definitions of the various matrices that appear in the following analysis are given in Appendix B. Defining $x = [z_o' \; z_1' \; z_2']'$, $y_i = [y_{oi}' \; \frac{1}{\sqrt{\epsilon_i}} y_{ii}']'$, $v_i = [v_{oi}' \; v_{ii}']'$, the system of equations (3.1)-(3.4) can be written in a composite form as

$$\dot{x} = Ax + \sum_{j=1}^{2} B_j u_j + Lw; \qquad x(0) = x_o \tag{3.5}$$

$$y_i = C_i x + v_i; \qquad i = 1,2. \tag{3.6}$$

$$E[x_o] = \bar{x}_o; \qquad E[(x_o - \bar{x}_o)(x_o - \bar{x}_o)'] = N \tag{3.7}$$

where $\dim x = n = n_o + n_1 + n_2$, $\dim y_i = P_i = P_{oi} + P_{ii}$

$$J_i = \frac{1}{2} \{x'(T)\Gamma_i x(T) + \int_o^T (x'Q_i x + u_i' R_i u_i) dt\}; \qquad i = 1,2. \tag{3.8}$$

Each decision maker is constrained to use only an n-dimensional compensator of the form

$$\dot{\hat{x}}_i = F_i \hat{x}_i + G_i [y_i - C_i \hat{x}_i] + H_i u_i; \qquad i = 1,2. \tag{3.9}$$

The decision makers are required to select the matrices F_i^*, G_i^*, H_i^*, the initial conditions $\hat{x}_i^*(0)$, and the closed-loop control laws $u_i^*(\hat{x}_i(t),t)$, such that

$$E[J_i(u_i^*, u_j^*) | X_i] \le E[J_i(u_i, u_j^*) | X_i]; \qquad i,j = 1,2; \quad i \ne j. \tag{3.10}$$

where X_i denotes a combination of $\hat{x}_i(t)$ and the a-priori information.

The pair of inequalities in (3.10) define the Nash equilibrium for the problem (3.5)-(3.9).

To solve the problem posed in equations (3.5)-(3.10), we need the following result which is a generalization of [35] for the nonzero-sum case.

Theorem 3.1: A sufficient condition for two closed-loop control laws (u_1^*, u_2^*) to be a Nash pair for the problem defined by (3.5)-(3.10) is that there exist real-valued functions $I_i(x,t)$ differentiable in each variable, which together with u_1^* and u_2^* satisfies for all $t \in [0,T]$ the following conditions:

Defining for all $t \in [0,T]$, the scalar functions $\tilde{S}_i$ by

$$\tilde{S}_i(x,u_1,u_2,t) = I_{it}(x,t) + I_{ix}(x,t)[Ax + \sum_{j=1}^{2} B_j u_j + Lw] + \frac{1}{2} x'Q_i x + \frac{1}{2} u_i'R_i u_i$$

$$\min_{u_i} E\{\tilde{S}_i(x,u_i,u_j^*,t) | X_i(t)\} = 0$$

$$E\{\tilde{S}_i(x,u_i^*,u_j^*,t) | X_i(t)\} = 0$$

$$I_i(x,T) = \frac{1}{2} x'\Gamma_i x$$

$$i,j = 1,2; \quad i \neq j.$$

Applying Theorem 3.1, the solution to the full problem (3.5)-(3.10) is given by

$$u_i^* = -R_i^{-1} B_i' K_i \hat{x}_i \tag{3.11a}$$

$$F_i^* = A - B_j R_j^{-1} B_j' K_j [I + (M_{jo} - M_{ji})(M_{oo} - M_{oi})^{-1}] \tag{3.11b}$$

$$G_i^* = M_{ii} C_i' V_i^{-1} \tag{3.11c}$$

$$H_i^* = B_i \tag{3.11d}$$

$$\hat{x}_i^*(0) = \bar{x}_o \tag{3.11e}$$

where K_i satisfies the coupled Riccati equation

$$\dot{K}_i + K_iA + A'K_i + Q_i - K_iS_iK_i - K_iS_jK_j - K_jS_jK_i = 0; \qquad K_i(T) = \Gamma_i. \tag{3.12}$$

M(t) is a symmetric nonnegative definite matrix defined as,

$$M(t) = E\{m(t)m'(t)\} \quad ; \qquad m(t) = \begin{bmatrix} x \\ x-\hat{x}_1 \\ x-\hat{x}_2 \end{bmatrix} \tag{3.13}$$

satisfying the differential equation

$$\dot{M} = FM + MF' + B \circledast B' ,$$

$$\text{with } M_{ij}(0) = \bar{x}_o\bar{x}_o' + N; \qquad i = j = 0$$

$$= N \qquad ; \qquad \text{elsewhere.} \tag{3.14}$$

The following relations can be readily derived:

$$E\{x(t)|X_i(t)\} = \hat{x}_i(t) \tag{3.15}$$

$$E\{(x(t)-\hat{x}_i(t))\hat{x}_i'(t)\} = 0 \tag{3.16}$$

$$E\{\hat{x}_j(t)|X_i(t)\} = [I + (M_{jo}-M_{ji})(M_{oo}-M_{oi})^{-1}]\hat{x}_i(t) \tag{3.17}$$

$$M_{io} = M_{oi}' = M_{ii}' = M_{ii} \tag{3.18}$$

$$I_i(x,t) = \frac{1}{2}\, x'K_ix + \frac{1}{2}\, b_i(t) \tag{3.19}$$

$$b_i(t) = tr\{\int_t^T (K_iS_iK_iM_{ii} + K_iS_jK_jM_{jo} + K_jS_jK_iM_{oj})d\tau\} \tag{3.20}$$

$$E\{J_i^*|X_i\} = \frac{1}{2}\,[x_i'(0)K_i(0)x_i(0) + tr\{M_{ii}(0)K_i(0)\} + b_i(0)]. \tag{3.21}$$

Notice that the optimal control gains are independent of the filter matrices and covariances; but the optimal filter matrices and covariances depend on

the control gains resulting in a "dual effect" which is optimized with respect to the given filter structure and the cost functionals.

The linear strategy (3.11a) is the unique Nash strategy for the above problem. Nonuniqueness does not arise because it is not possible to express $\hat{x}_i$ at time t, in terms of the values of $\hat{x}_i$ from 0 to t, due to the presence of white-noise-corrupted measurements (3.6) [27].

The following assumptions are made throughout.

Assumption a: Re $\lambda(A_{ii}) < 0$; $i = 1,2$.

Assumption b: The triple (A_{ii}, B_{ii}, C_{ii}) is controllable-observable.

From the solution obtained above, it is clear that the optimal finite-dimensional compensators are not Kalman filters, and hence the earlier results [30-34] on filtering and control of linear stochastic singularly perturbed systems do not apply here. A number of important questions now arise: What is the limiting structure of these finite-dimensional compensators as the small parameters go to zero? Does the full order compensator decompose into a number of decoupled low-order compensators? Does the resulting limiting structure offer any computational and/or implementational advantages? Is it possible to obtain a near-equilibrium solution based on the solution of low-order problems as in the deterministic case [15,16]?

It is our intention here to answer the above questions. Specifically we shall show that the multimodel solution is the asymptotic limit of the exact solution as the small parameters go to zero. To obtain the multimodel solution, we first need to derive the simplified model used by each decision maker.

3.3. Multimodel Generation

DMi arrives at his simplified model by neglecting the dynamics of the j-th fast subsystem and the weak interactions between the two fast subsystems, i.e., by setting $\varepsilon_j = 0$ on the left hand side of (3.1) and $\varepsilon_{11} = \varepsilon_{22} = 0$ in (3.1). The steady state dynamics of the j-th fast subsystem is then given by the algebraic equation

$$\bar{z}_j(t) = -A_{jj}^{-1}(A_{jo}z_o^{(i)} + B_{jj}u_j + \sqrt{\varepsilon_j}\; L_j w). \tag{3.22}$$

The above expression for $\bar{z}_j(t)$ has been shown to be valid as input to slow systems [31]. Therefore, substituting (3.22) in (3.1), (3.2) results in the following simplified model for the ith decision maker,

$$\dot{z}_o^{(i)} = A_o^{(i)} z_o^{(i)} + A_{oi}z_i^{(i)} + B_{js}u_j + B_{oi}u_i + L_o^{(i)}w; \quad z_o^{(i)}(0) = z_{oo}. \tag{3.23a}$$

$$\varepsilon_i \dot{z}_i^{(i)} = A_{io}z_o^{(i)} + A_{ii}z_i^{(i)} + B_{ii}u_i + \sqrt{\varepsilon_i}\; L_i w; \qquad z_i^{(i)}(0) = z_{io}. \tag{3.23b}$$

The observation vectors for the two players are given by

$$y_i^{(i)} = \begin{bmatrix} C_{oi} & 0 \\ 0 & C_{ii} \end{bmatrix} \begin{bmatrix} z_o^{(i)} \\ z_i^{(i)} \end{bmatrix} + v_i \tag{3.24a}$$

$$y_j^{(i)} = C_{js}z_o^{(i)} + D_{js}u_j + v_{js}. \tag{3.24b}$$

Notice that in the above simplified model used by DMi, the two decision makers do not interact at the fast subsystem level, but interact only at the slow subsystem level. Therefore, to obtain the multimodel solution, DMi needs only to know the parameters associated with the model (3.23), (3.24) ; an

exact knowledge of the full model (3.1), (3.2) is not required. The multi-model solution is then obtained by solving three low-order problems: two independent stochastic control problems for each decision maker at the fast subsystem level; and a constrained stochastic Nash game at the slow subsystem level.

3.4. Weak Limit of the Fast Stochastic Variable

Before we formulate the low-order problems, we would like to establish the "weak" limit (limit in the sense of distributions) of the fast stochastic variable which will be shown to be the valid limit for substitution into the cost functionals. The formal white noise limit (3.22) is not valid for substitution into the cost functionals since it gives rise to some ill-defined terms like the integral of the variance of white noise [31].

The following results are needed:

Lemma 3.1: Let $f(t)$ be a function satisfying the following conditions

i) $f(t) \geq 0$ for all t

ii) $\int_{-\infty}^{\infty} f(t)dt = 1.$

Then the following distribution convergence is obtained,

$$\lim_{\mu \to 0} \frac{1}{\mu} f(t/\mu) = \delta(t).$$

Lemma 3.2: Let $z(t) = \frac{1}{\sqrt{\mu}} \int_0^t e^{A(t-\tau)/\mu} L\, d\bar{w}$, where $\bar{w}$ is a Wiener process with $E\{d\bar{w}(\tau_1)d\bar{w}(\tau_2)\} = W\delta(\tau_1-\tau_2)d\tau$. Then, $\lim_{\mu \to 0} z(t) = \tilde{w}$ "weakly" for each $t > 0$, where $\tilde{w}$ is a constant Gaussian random vector with mean zero and variance $\tilde{W}$ which satisfies the equation

$$A\tilde{W} + \tilde{W}A' + LWL' = 0.$$

Setting $\varepsilon_{11} = \varepsilon_{22} = 0$ we rewrite equations (3.1b) as

$$\varepsilon_i dz_i = A_{io} z_o dt + A_{ii} z_i dt + B_{ii} u_i dt + \sqrt{\varepsilon_i}\, L_i d\bar{w} \tag{3.25}$$

where $\bar{w}$ is a Wiener process such that $\dot{\bar{w}} = w$. The integral representation of equation (3.25) can be written as

$$z_i(t) = e^{A_{ii}t/\varepsilon_i} z_{io} + \frac{1}{\varepsilon_i}\int_o^t e^{A_{ii}(t-\tau)/\varepsilon_i}(A_{io}z_o + B_{ii}u_i)d\tau$$

$$+ \frac{1}{\sqrt{\varepsilon_i}}\int_o^t e^{A_{ii}(t-\tau)/\varepsilon_i} L_i d\bar{w}(\tau). \tag{3.26}$$

A straightforward application of Lemmas 3.1 and 3.2 yields the following "weak" limit

$$\tilde{z}_i(t) = \lim_{\varepsilon_i \to 0} z_i(t) = -A_{ii}^{-1}(A_{io}z_o + B_{ii}u_i) + \tilde{w}_i, \tag{3.27}$$

where $\tilde{w}_i$ is a constant Gaussian random vector with mean zero and variance $\tilde{W}_i$ which satisfies the equation

$$A_{ii}\tilde{W}_i + \tilde{W}_i A_{ii}' + L_i W L_i' = 0. \tag{3.28}$$

3.5. Slow Subproblem

The slow subproblem is formulated by setting $\varepsilon_{11} = \varepsilon_{22} = 0$ and $\varepsilon_1 = \varepsilon_2 = 0$ on the left hand side of (3.1). The formal white noise limit given by (3.22) is substituted into the state and observation equations (3.1) and (3.2); and the weak limit (3.2) is substituted into the cost functionals (3.4). This gives

$$\dot{z}_{os} = A_s z_{os} + \sum_{j=1}^{2} B_{js} u_{js} + L_{os} w; \qquad z_{os}(0) = z_{oo} . \tag{3.29}$$

$$y_{is} = C_{is} z_{os} + D_{is} u_{is} + v_{is}; \qquad i = 1,2. \tag{3.30}$$

Each player is constrained to use only an n-dimensional compensator of the form

$$\dot{\hat{z}}_{si} = F_{is} \hat{z}_{si} + G_{is} [y_{is} - C_{is} \hat{z}_{si} - D_{is} u_{is}] + H_{is} u_{is} . \tag{3.31}$$

The expected values of the cost functionals are given by

$$E[J_{is} | Z_{si}] = \frac{1}{2} E\{z'_{os}(T) \bar{\Gamma}_{oi} z_{os}(T) + \int_o^T (z'_{os} Q_{ois} Z_{os} + 2z'_{os} Q_{is} u_{is} + u'_{is} R_{is} u_{is}) dt | Z_{si}\} + \frac{T}{2} \operatorname{tr}\{\bar{Q}_i \tilde{W}_i\}. \tag{3.32}$$

The decision makers select the matrices F^*_{is}, G^*_{is}, H^*_{is}; the initial conditions $\hat{z}^*_{si}(0)$; and the closed-loop control laws $u^*_{is}(\hat{z}_{si}(t),t)$ such that

$$E\{J_{is}(u^*_{is}, u^*_{js}) | Z_{si}\} \le E\{J_{is}(u_{is}, u^*_{js}) | Z_{si}\}; \qquad i,j = 1,2; \quad i \neq j. \tag{3.33}$$

Applying Theorem 3.1, the equilibrium solution is found as

$$u^*_{is} = -R^{-1}_{is} [B'_{is} K_{is} + Q'_{is}] \hat{z}_{si} \tag{3.34a}$$

$$F^*_{is} = A_s - B_{js} R^{-1}_{js} (B'_{js} K_{js} + Q'_{js}) [I + (\bar{M}_{jo} - \bar{M}_{ji})(\bar{M}_{oo} - \bar{M}_{oi})^{-1}] \tag{3.34b}$$

$$G^*_{is} = (\bar{M}_{ii} C'_{is} - L_{os} W \bar{L}'_i) V^{-1}_{is} \tag{3.34c}$$

$$H^*_{is} = B_{is} \tag{3.34d}$$

$$\hat{z}^*_{si}(0) = \bar{z}_{oo} \tag{3.34e}$$

where K_{is} is the solution of the coupled Riccati equation

$$\dot{K}_{is} + K_{is}\tilde{A}_s + \tilde{A}_s' K_{is} + (Q_{ois} - Q_{is}R_{is}^{-1}Q_{is}') - K_{is}S_{is}K_{is} - K_{is}S_{js}K_{js}$$

$$- K_{js}S_{js}K_{is} = 0; \qquad K_{is}(T) = \bar{\Gamma}_{oi}. \tag{3.35}$$

$\bar{M}(t)$ is a symmetric nonnegative definite matrix defined as,

$$\bar{M}(t) = E\{\bar{m}(t)\bar{m}'(t)\} \quad ; \qquad \bar{m}(t) = \begin{bmatrix} z_{os} \\ z_{os} - \hat{z}_{s1} \\ z_{os} - \hat{z}_{s2} \end{bmatrix} \tag{3.36}$$

satisfying the differential equation,

$$\dot{\bar{M}} = F_s\bar{M} + \bar{M}F_s' + B_s \circledast_s B_s'$$

with

$$\bar{M}_{ij}(0) = \bar{z}_{oo}\bar{z}_{oo}' + N_{oo} \; ; \qquad i = j = 0.$$

$$= N_{oo} \qquad ; \qquad \text{else.} \tag{3.37}$$

The expected value of the optimal cost is obtained as

$$E\{J_{is}^* | Z_{si}\} = \frac{1}{2}\,[\hat{z}_{si}'(0)K_{is}(0)\hat{z}_{si}(0) + tr\{\bar{M}_{ii}(0)K_{is}(0)\} + b_{is}(0)]$$

$$+ \frac{T}{2}\, tr\; [\bar{Q}_i\tilde{W}_i] \quad , \tag{3.38}$$

where

$$b_{is}(t) = tr\{\int_t^T \{[Q_{is} + K_{is}B_{is}]R_{is}^{-1}[Q_{is}' + B_{is}'K_{is}]\bar{M}_{ii} + K_{is}B_{js}R_{js}^{-1}(B_{js}'K_{js}$$

$$+ Q_{js}')\bar{M}_{jo} + (Q_{js} + K_{js}B_{js})R_{js}^{-1}B_{js}'K_{is}\bar{M}_{oj}\}d\tau\}. \tag{3.39}$$

3.6. Fast Subproblems

The fast subproblems are "local" problems for each decision maker. These are stochastic control problems because the decision makers do not interact at the fast subsystem level. Assuming that the slow variables are constant during the fast transients, we obtain

$$\varepsilon_i \dot{z}_{if} = A_{ii} z_{if} + B_{ii} u_{if} + \sqrt{\varepsilon_i}\, L_i w \tag{3.40}$$

$$y_{iif} = C_{ii} z_{if} + \sqrt{\varepsilon_i}\, v_{ii} \tag{3.41}$$

$$J_{if} = \frac{1}{2} \{\varepsilon_i z'_{if}(T) \bar{\Gamma}_i z_{if}(T) + \int_0^T (z'_{if} \bar{Q}_i z_{if} + u'_{if} R_i u_{if}) dt\}. \tag{3.42}$$

The optimal u^*_{if} minimizing $E(J_{if})$ is obtained by applying the separation principle, so that

$$u^*_{if} = -R_i^{-1} B'_{ii} K_{if} \hat{z}_{if} \tag{3.43}$$

where K_{if} satisfies the Riccati equation

$$\varepsilon_i \dot{K}_{if} = -K_{if} A_{ii} - A'_{ii} K_{if} - \bar{Q}_i + K_{if} S_{ii} K_{if} \ ; \qquad K_{if}(T) = \bar{\Gamma}_i; \tag{3.44}$$

$\hat{z}_{if}$ is the output of the Kalman filter given by

$$\varepsilon_i \dot{\hat{z}}_{if} = A_{ii}\hat{z}_{if} + B_{ii}u^*_{if} + P_{if}C'_{ii}V^{-1}_{ii}[y_{iif} - C_{ii}\hat{z}_{if}]; \quad \hat{z}_{if}(0) = \bar{z}_{io}$$

$$= (A_{ii} - S_{ii}K_{if})\hat{z}_{if} + P_{if}C'_{ii}V^{-1}_{ii}[y_{ii} + C_{ii}A^{-1}_{ii}(A_{io}\hat{z}_{si} + B_{ii}u^*_{is}) - C_{ii}\hat{z}_{if}]$$

$$= (A_{ii} - S_{ii}K_{if} - P_{if}T_{ii})\hat{z}_{if} + P_{if}C'_{ii}V^{-1}_{ii}[y_{ii} + C_{ii}A^{-1}_{ii}\{A_{io} - B_{ii}R^{-1}_{is}(B'_{is}K_{is} + Q'_{is})\}\hat{z}_{si}]. \tag{3.45}$$

P_{if} is the error covariance of $\hat{z}_{if}$ satisfying

$$\varepsilon_i \dot{P}_{if} = P_{if}A'_{ii} + A_{ii}P_{if} + L_iWL'_i - P_{if}T_{ii}P_{if} \ ; \quad P_{if}(0) = N_{ii}. \tag{3.46}$$

Under the Assumptions (a) and (b), the limiting behavior of $\hat{z}_{if}$, P_{if}, K_{if}, and u^*_{if} as $\varepsilon_i \to 0$ has been considered in [31] and is summarized below:

$$u^*_{if} = \bar{u}^*_{if} + 0(\varepsilon_i^{1/2}) \tag{3.47a}$$

$$\hat{z}_{if} = \hat{\bar{z}}_{if} + 0(\varepsilon_i^{1/2}) \tag{3.47b}$$

$$K_{if} = \bar{K}_{if} + 0(\varepsilon_i) \tag{3.47c}$$

$$P_{if} = \bar{P}_{if} + 0(\varepsilon_i) \tag{3.47d}$$

where $\bar{P}_{if}$, $\bar{K}_{if}$, $\hat{\bar{z}}_{if}$, and $\bar{u}^*_{if}$ satisfy

$$\bar{P}_{if}A'_{ii} + A_{ii}\bar{P}_{if} + L_iWL'_i - \bar{P}_{if}T_{ii}\bar{P}_{if} = 0 \tag{3.48a}$$

$$\bar{K}_{if}A_{ii} + A'_{ii}\bar{K}_{if} + \bar{Q}_i - \bar{K}_{if}S_{ii}\bar{K}_{if} = 0 \tag{3.48b}$$

$$\varepsilon_i \dot{\hat{\bar{z}}}_{if} = (A_{ii} - S_{ii}\bar{K}_{if} - \bar{P}_{if}T_{ii})\hat{\bar{z}}_{if} + \bar{P}_{if}C'_{ii}V^{-1}_{ii}[y_{ii} + C_{ii}A^{-1}_{ii}\{A_{io} - B_{ii}R^{-1}_{is}(B'_{is}K_{is} + Q'_{is})\}\hat{z}_{si}] \tag{3.48c}$$

$$\bar{u}^*_{if} = -R^{-1}_iB'_{ii}\bar{K}_{if}\hat{\bar{z}}_{if}. \tag{3.48d}$$

The expected value of the optimal cost is given by

$$E(J^*_{if}) = \int_0^T tr\{[\frac{1}{2}\bar{Q}_i + K_{if}P_{if}T_{ii}]P_{if}\}dt + \frac{\varepsilon_i}{2} tr[P_{if}(T)\bar{\Gamma}_i] + \frac{\varepsilon_i}{2}\bar{z}'_{io}K_{if}(0)\bar{z}_{io}. \tag{3.49}$$

In the limit as $\varepsilon_i \to 0$ this reduces to

$$E(J^*_{if}) = T\ tr\{[\frac{1}{2}\bar{Q}_i + \bar{K}_{if}\bar{P}_{if}T_{ii}]\bar{P}_{if}\}. \tag{3.50}$$

The approximations obtained from equations (3.47) and (3.48) are valid only on a subinterval $[t_1,t_2] \subset (0,T)$ because the "boundary-layer" terms have been neglected.

3.7. Limiting Behavior of the Optimal Solution

The multimodel strategy pair used by the decision makers is given by

$$u^*_{im} = u^*_{is} + \bar{u}^*_{if} = -R^{-1}_{is}[B'_{is}K_{is} + Q'_{is}]\hat{z}_{si} - R^{-1}_i B'_{ii}\bar{K}_{if}\hat{\bar{z}}_{if}; \quad i = 1,2 \tag{3.51}$$

where $\hat{z}_{si}$ and $\hat{\bar{z}}_{if}$ are the states of the n_o- and n_i-dimensional compensators given by (3.31) and (3.48c), respectively.

We shall now examine the limiting behavior of the exact solution (3.11)-(3.14). For the sake of brevity, the detailed manipulations involved in taking the limit of matrix equations as $\|\varepsilon_i\| \to 0$ are omitted.

Let the solution of (3.12), K_i, be of the form

$$K_i = \begin{bmatrix} K_{00}^{(i)}(\varepsilon) & \varepsilon_1 K_{01}^{(i)}(\varepsilon) & \varepsilon_2 K_{02}^{(i)}(\varepsilon) \\ \varepsilon_1 K_{01}^{(i)'}(\varepsilon) & \varepsilon_1 K_{11}^{(i)}(\varepsilon) & \sqrt{\varepsilon_1\varepsilon_2} K_{12}^{(i)}(\varepsilon) \\ \varepsilon_2 K_{02}^{(i)'}(\varepsilon) & \sqrt{\varepsilon_1\varepsilon_2} K_{12}^{(i)'}(\varepsilon) & \varepsilon_2 K_{22}^{(i)}(\varepsilon) \end{bmatrix}; \; i = 1,2. \tag{3.52}$$

Substituting this in (3.12) and taking the limit as $\|\varepsilon\| \to 0$, it can be shown that

$$K_{00}^{(i)}(\varepsilon) = K_{is} + O(\|\varepsilon\|)$$

$$K_{0i}^{(i)}(\varepsilon) = K_{is}\hat{E}_i - \tilde{E}_i + O(\|\varepsilon\|)$$

$$K_{0j}^{(i)}(\varepsilon) = K_{is}\hat{E}_j + O(\|\varepsilon\|)$$

$$K_{ii}^{(i)}(\varepsilon) = \bar{K}_{if} + O(\|\varepsilon\|)$$

$$K_{ij}^{(i)}(\varepsilon) = O(\|\varepsilon\|)$$

$$K_{jj}^{(i)}(\varepsilon) = O(\|\varepsilon\|)$$

where

$$\hat{E}_i = (\tilde{S}_{0i}\bar{K}_{if} - A_{0i})(A_{ii} - S_{ii}\bar{K}_{if})^{-1}$$

$$\tilde{E}_i = A'_{i0}\bar{K}_{if}(A_{ii} - S_{ii}\bar{K}_{if})^{-1} \tag{3.53}$$

Let the solution of (3.14), M(t), be of the form

$$M(\varepsilon) = \begin{bmatrix} M_{00}(\varepsilon) & M_{11}(\varepsilon) & M_{22}(\varepsilon) \\ M_{11}(\varepsilon) & M_{11}(\varepsilon) & M_{12}(\varepsilon) \\ M_{22}(\varepsilon) & M'_{12}(\varepsilon) & M_{22}(\varepsilon) \end{bmatrix} \tag{3.54a}$$

where each block is of the form

$$M_{00}(\varepsilon) = \begin{bmatrix} M_{00}^{00}(\varepsilon) & \sqrt{\varepsilon_1}M_{00}^{01}(\varepsilon) & \sqrt{\varepsilon_2}M_{00}^{02}(\varepsilon) \\ \sqrt{\varepsilon_1}M_{00}^{01'}(\varepsilon) & M_{00}^{11}(\varepsilon) & M_{00}^{12}(\varepsilon) \\ \sqrt{\varepsilon_2}M_{00}^{02'}(\varepsilon) & M_{00}^{12'}(\varepsilon) & M_{00}^{22}(\varepsilon) \end{bmatrix},$$

$$M_{ii}(\varepsilon) = \begin{bmatrix} M_{ii}^{00}(\varepsilon) & \sqrt{\varepsilon_1}M_{ii}^{01}(\varepsilon) & \sqrt{\varepsilon_2}M_{ii}^{02}(\varepsilon) \\ \sqrt{\varepsilon_1}M_{ii}^{01'}(\varepsilon) & M_{ii}^{11}(\varepsilon) & M_{ii}^{12}(\varepsilon) \\ \sqrt{\varepsilon_2}M_{ii}^{02'}(\varepsilon) & M_{ii}^{12'}(\varepsilon) & M_{ii}^{22}(\varepsilon) \end{bmatrix},$$

$$M_{12}(\varepsilon) = \begin{bmatrix} M_{12}^{00}(\varepsilon) & \sqrt{\varepsilon_1}M_{12}^{01}(\varepsilon) & \sqrt{\varepsilon_2}M_{12}^{02}(\varepsilon) \\ \sqrt{\varepsilon_1}M_{12}^{10}(\varepsilon) & M_{12}^{11}(\varepsilon) & M_{12}^{12}(\varepsilon) \\ \sqrt{\varepsilon_2}M_{12}^{20}(\varepsilon) & M_{12}^{21}(\varepsilon) & M_{12}^{22}(\varepsilon) \end{bmatrix} \tag{3.54b}$$

Substituting (3.54) in (3.14) and taking the limit as $\|\varepsilon\| \to 0$, it can be shown that,

$$\begin{aligned} M_{ii}^{00}(\varepsilon) &= \bar{M}_{ii} + 0(\|\varepsilon\|) \\ M_{ii}^{ii}(\varepsilon) &= \bar{P}_{if} + 0(\|\varepsilon\|) \\ M_{ii}^{12}(\varepsilon) &= 0(\|\varepsilon\|) \\ M_{00}^{00}(\varepsilon) &= \bar{M}_{00} + 0(\|\varepsilon\|) \\ M_{00}^{12}(\varepsilon) &= 0(\|\varepsilon\|) \\ M_{12}^{00}(\varepsilon) &= \bar{M}_{12} + 0(\|\varepsilon\|) \\ M_{12}^{ii}(\varepsilon) &= 0(\|\varepsilon\|) \\ M_{12}^{ij}(\varepsilon) &= 0(\|\varepsilon\|) . \end{aligned} \tag{3.55a}$$

The remaining matrices, which we do not need explicitly, satisfy the following set of equations:

$$A_{i0}M_{jj}^{0i}(0)+A_{ii}M_{jj}^{ii}(0)+M_{jj}^{0i'}(0)A_{i0}'+M_{jj}^{ii}(0)A_{ii}'+L_iWL_i'=0$$

$$A_{0i}M_{jj}^{ii}(0)+M_{jj}^{0i}(0)A_{ii}'+\bar{M}_{jj}A_{i0}'+L_0WL_i'=0$$

$$A_{0i}M_{jj}^{ii}(0)+M_{12}^{0i}(0)A_{ii}'+\bar{M}_{12}A_{i0}'+L_0WL_i'=0$$

$$A_{0i}M_{00}^{ii}(0)+M_{00}^{0i}(0)A_{ii}'+\bar{M}_{00}A_{i0}'+L_0WL_i'=0$$

$$A_{i0}M_{00}^{0i}(0)+A_{ii}M_{00}^{ii}(0)+M_{00}^{0i'}(0)A_{i0}'+M_{00}^{ii}(0)A_{ii}'+L_iWL_i'=0$$

$$A_{0i}\bar{P}_{if}+M_{ii}^{0i}(0)A_{ii}'+\bar{M}_{ii}A_{i0}'+L_0WL_i'-(\bar{M}_{ii}\bar{C}_{0i}'+M_{ii}^{0i}(0)\bar{C}_i')V_i^{-1}\bar{C}_i\bar{P}_{if}=0$$

$$A_{0i}\bar{P}_{if}+M_{12}^{0i}(0)A_{ii}'+\bar{M}_{12}A_{i0}'+L_0WL_i'-(\bar{M}_{12}\bar{C}_{0j}'+M_{12}^{0i}(0)\bar{C}_i')V_{ji}^{-1}\bar{C}_i\bar{P}_{if}=0\,. \qquad (3.55b)$$

The limiting solutions given by (3.53) and (3.55) are valid only on a subinterval $[t_1,t_2]\subset[0,T]$, because the boundary-layer terms have been neglected.

Write the closed-loop system as,

$$\dot{x} = Ax - S_1K_1\hat{x}_1 - S_2K_2\hat{x}_2 + Lw \qquad (3.56a)$$

$$\dot{\hat{x}}_i = (F_i - S_iK_i)\hat{x}_i + G_i\nu_i\ ;\ \nu_i = y_i - C_i\hat{x}_i;\ i=1,2. \qquad (3.56b)$$

Define

$$\tilde{x}_i = \begin{bmatrix} \hat{z}_{i1} \\ \hat{z}_{i2} \end{bmatrix}\ ;\quad i=0,1,2\ ; \qquad (3.57)$$

then (3.56b) can be rewritten as,

$$\dot{\tilde{x}}_0 = \bar{A}_{00}\tilde{x}_0 + \bar{A}_{01}\tilde{x}_1 + \bar{A}_{02}\tilde{x}_2 + \bar{G}_0\nu \qquad (3.58a)$$

$$\varepsilon_1\dot{\tilde{x}}_1 = \bar{A}_{10}\tilde{x}_0 + \bar{A}_{11}\tilde{x}_1 + \varepsilon_{11}\bar{A}_{12}\tilde{x}_2 + \sqrt{\varepsilon_1}\bar{G}_1\ \nu \qquad (3.58b)$$

$$\varepsilon_2\dot{\tilde{x}}_2 = \bar{A}_{20}\tilde{x}_0 + \varepsilon_{22}\bar{A}_{21}\tilde{x}_1 + \bar{A}_{22}\tilde{x}_2 + \sqrt{\varepsilon_2}\bar{G}_2\nu. \qquad (3.58c)$$

The form of the compensator equations (3.58) is identical to the form of the state equations (3.1). This form permits easier manipulations to obtain their limiting behavior.

Now, we transform the equations (3.58), in order to separate its slow and fast components. The transformation and its inverse are:

$$\begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} I_0 - \sqrt{\varepsilon_1} N_1 T_1 - \sqrt{\varepsilon_2} N_2 T_2 & -\sqrt{\varepsilon_1} N_1 & -\sqrt{\varepsilon_2} N_2 \\ T_1 & I_1 & 0 \\ T_2 & 0 & I_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_0 \\ \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \tag{3.59a}$$

$$\begin{bmatrix} \tilde{x}_0 \\ \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} I_0 & \sqrt{\varepsilon_1} N_1 & \sqrt{\varepsilon_2} N_2 \\ -T_1 & I_1 - \sqrt{\varepsilon_1} T_1 N_1 & -\sqrt{\varepsilon_2} T_1 N_2 \\ -T_2 & -\sqrt{\varepsilon_1} T_2 N_1 & I_2 - \sqrt{\varepsilon_2} T_2 N_2 \end{bmatrix} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \end{bmatrix} \tag{3.59b}$$

where,

$$\varepsilon_i \dot{T}_i = \bar{A}_{ii} T_i - \bar{A}_{i0} - \varepsilon_i T_i (\bar{A}_{00} - \bar{A}_{01} T_1 - \bar{A}_{02} T_2) + \varepsilon_{ii} \bar{A}_{ij} T_j \tag{3.60a}$$

$$\begin{aligned} \varepsilon_i \dot{N}_i = & -N_i (\bar{A}_{ii} - \bar{G}_i \bar{C}_i - \sqrt{\varepsilon_i} T_i \bar{G}_0 \bar{C}_i) - \sqrt{\varepsilon_j} N_j T_j (\sqrt{\varepsilon_i} \bar{A}_{0i} - \bar{G}_0 \bar{C}_i) \\ & + \sqrt{\varepsilon_i} \bar{A}_{0i} - \bar{G}_0 \bar{C}_i + \varepsilon_i (\bar{A}_{00} - \bar{A}_{01} T_1 - \bar{A}_{02} T_2) N_i - \sqrt{\frac{\varepsilon_i}{\varepsilon_j}}\, \varepsilon_{ii} N_j \bar{A}_{ji}. \quad i,j=1,2; i \neq j \end{aligned} \tag{3.60b}$$

Transforming (3.58) using (3.59) and (3.60) results in;

$$\begin{aligned} \dot{\eta}_0 = & [\bar{A}_{00} - \bar{A}_{01} T_1 - \bar{A}_{02} T_2] \eta - [N_1 (\frac{1}{\sqrt{\varepsilon_1}} \bar{G}_{11} \bar{C}_1 + T_1 \bar{G}_0 \bar{C}_1) + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\, N_2 T_2 \bar{G}_0 \bar{C}_1 - \frac{1}{\sqrt{\varepsilon_1}} \bar{G}_{10} \bar{C}_1 \\ & + \sqrt{\varepsilon_1} N_1 T_1 \bar{A}_{01}] \eta_1 - [N_2 (\frac{1}{\sqrt{\varepsilon_2}} \bar{G}_2 \bar{C}_2 + T_2 \bar{G}_0 \bar{C}_2) + \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\, N_1 T_1 \bar{G}_0 \bar{C}_2 - \frac{1}{\sqrt{\varepsilon_2}} \bar{G}_0 \bar{C}_2 \\ & + \sqrt{\varepsilon_2} N_2 T_2 \bar{A}_{02}] \eta_2 + [(I_0 - \sqrt{\varepsilon_1} N_1 T_1 - \sqrt{\varepsilon_2} N_2 T_2) \bar{G}_0 - N_1 \bar{G}_1 - N_2 \bar{G}_2] \nu \end{aligned} \tag{3.61a}$$

$$\varepsilon_1 \dot{\eta}_1 = [\varepsilon_1 T_1 \bar{A}_{01} + \bar{A}_{11}] \eta_1 + [\varepsilon_1 T_1 \bar{A}_{02} + \varepsilon_{11} \bar{A}_{12}] \eta_2 + [\varepsilon_1 T_1 \bar{G}_0 + \sqrt{\varepsilon_1} \bar{G}_1] \nu \tag{3.61b}$$

$$\varepsilon_2 \dot{\eta}_2 = [\varepsilon_2 T_2 \bar{A}_{01} + \varepsilon_{22} \bar{A}_{21}] \eta_1 + [\varepsilon_2 T_2 \bar{A}_{02} + \bar{A}_{22}] \eta_2 + [\varepsilon_2 T_2 \bar{G}_0 + \sqrt{\varepsilon_2} \bar{G}_2] \nu . \tag{3.61c}$$

It can be shown that the limiting solution of (3.61) is

$$\eta_0 = \begin{bmatrix} \eta_0^{(1)} \\ \eta_0^{(2)} \end{bmatrix} = \begin{bmatrix} \hat{z}_{s1} \\ \hat{z}_{s2} \end{bmatrix} + O(\|\varepsilon\|^{1/2}) \tag{3.62a}$$

$$\eta_1 = \begin{bmatrix} \eta_1^{(1)} \\ \eta_1^{(2)} \end{bmatrix} = \begin{bmatrix} \hat{\bar{z}}_{1f} \\ \bar{\eta}_1^{(2)} \end{bmatrix} + O(\|\varepsilon\|^{1/2}) \tag{3.62b}$$

$$\eta_2 = \begin{bmatrix} \eta_2^{(1)} \\ \eta_2^{(2)} \end{bmatrix} = \begin{bmatrix} \bar{\eta}_2^{(1)} \\ \hat{\bar{z}}_{2f} \end{bmatrix} + O(\|\varepsilon\|^{1/2}) \quad , \tag{3.62c}$$

where

$$\varepsilon_i \dot{\bar{\eta}}_i^{(j)} = (A_{ii} - B_{ii}R_i^{-1}B_{ii}'\bar{K}_{if})\bar{\eta}_i^{(j)}; \; \bar{\eta}_i^{(j)}(0) = \bar{z}_{i0} \tag{3.63}$$

The limiting solution of η_0 is just the compensator of the slow subproblem; and the limiting solution of one component of η_i is the Kalman filter of the fast subproblem. The other component of η_i, which is the estimate of the ith fast state by the jth decision maker, tends to a filter based on the a-priori information which is all the jth decision maker knows about the ith fast subsystem. This estimate is of no use to him since his near-optimal strategy given below does not need this information.

The equilibrium strategies are approximated as,

$$u_i^* = -R_i^{-1}B_i'K_i\hat{x}_i = -R_i^{-1}B_{ii}'\bar{K}_{if}\hat{\bar{z}}_{if} - R_{is}^{-1}[B_{is}'K_{is} + Q_{is}']\hat{z}_{si} + O(\|\varepsilon\|^{1/2}) = u_{im}^* + O(\|\varepsilon\|^{1/2});$$
$$i = 1,2. \tag{3.64}$$

The optimal expected values of the performance indices are approximated as,

$$E\{J_i^*|X_i\} = \frac{1}{2}[\hat{x}_i'(0)K_i(0)\hat{x}_i(0) + tr\{M_{ii}(0)K_i(0)\} + b_i(0)]$$

$$= T\ tr[\bar{K}_{if}\bar{P}_{if}C_{ii}'V_{ii}^{-1}C_{ii}'\bar{P}_{if}] + \frac{1}{2}T\ tr[\bar{Q}_i(\bar{P}_{if}+\tilde{W}_i)]$$

$$+\frac{1}{2}[\hat{z}_{si}'(0)K_{is}(0)\hat{z}_{si}(0) + tr\{\bar{M}_{ii}(0)K_{is}(0)\} + b_{is}(0)] + 0(\|\varepsilon\|)$$

$$= E\{J_{is}^*|Z_{si}\} + E\{\bar{J}_{if}^*\} + 0(\|\varepsilon\|); \qquad i = 1,2. \tag{3.65}$$

Equations (3.64) and (3.65) are obtained by substituting the limiting values of K_i and M. To get (3.64), $\hat{x}_i$ also had to be transformed using a transformation similar to (3.59).

The multimodel nature of the problem is apparent from the form of the near-optimal strategies (3.64), which suggests that the ith decision maker needs only to model the dynamics of his own fast subsystem and the common slow subsystem.

The structure of the near-optimal scheme is similar to that of the deterministic problem treated in [15,16], in the sense that the fast subproblems are control problems different for the two decision makers and the slow game problem is common to both the decision makers. This is essentially due to the fact that in both cases the fast subsystems are weakly-coupled and are controlled by a single decision maker. In situations when this is not true, the near-optimal solution will be quite different as has been demonstrated for deterministic problems in Chapter 2.

The overall near-optimal filtering-control scheme is depicted in Fig. 3.1. The hierarchical nature of the filter implementation, wherein the estimate of the slow filter is one of the driving inputs to the fast filter, can be seen from the figure. This arises naturally due to the fact that

the innovations process driving the fast filter needs the "fast" output which is generated from the actual output by subtracting out its "slow" part formed from the slow estimate. This fact has been pointed out in [34] for the single parameter control problem.

3.8. Conclusions

A decentralized filtering and control scheme has been presented for two decision makers controlling a large scale system. It is shown that in order to obtain near-equilibrium Nash strategies, the decision makers need only solve two decoupled low-order problems: a stochastic control problem in the fast time-scale at their "local" level, and a joint slow game problem with finite-dimensional state estimators. This leads directly to a multimodel situation wherein each decision maker needs to model only his local dynamics and some aggregate dynamics of the rest of the system. The advantages of using the proposed scheme are apparent. The decoupling of solutions at the subsystem level would result in considerable computational saving. Also since the near-optimal strategies need only decentralized "state estimates," each decision maker needs to construct only two filters of dimensions n_0 and n_i, respectively, instead of constructing one filter of dimension $n_0+n_1+n_2$ as required by the equilibrium solution. This would result in lower implementation costs.

It is to be noted that the problem addressed in this chapter is quite different from the earlier problems on filtering and control of stochastic singularly perturbed systems. The earlier work focused on appropriately characterizing the limiting behavior of the fast variable in

the presence of white noise to obtain well-posed lower order problems. The high-order optimal singularly perturbed Kalman filter was shown to decompose into two low-order Kalman filters in the slow and fast time-scales in the limit as $\varepsilon \to 0$. The problem with multiple decision makers possessing differing observations under a multimodel situation has been addressed here for the first time. Since the estimators for this problem are not Kalman filters, the earlier results could not be applied here. Therefore we had to examine the limiting behavior of the particular estimator structure adopted for the optimal solution. The result shows that in the slow time-scale the estimator retains the same structure as the optimal, but in the fast time-scale it turns out to be a Kalman filter. Furthermore, we have established the "weak" convergence of the fast variable which is shown to be the valid limit for substitution in the cost functionals; a fact which had not been established so far.

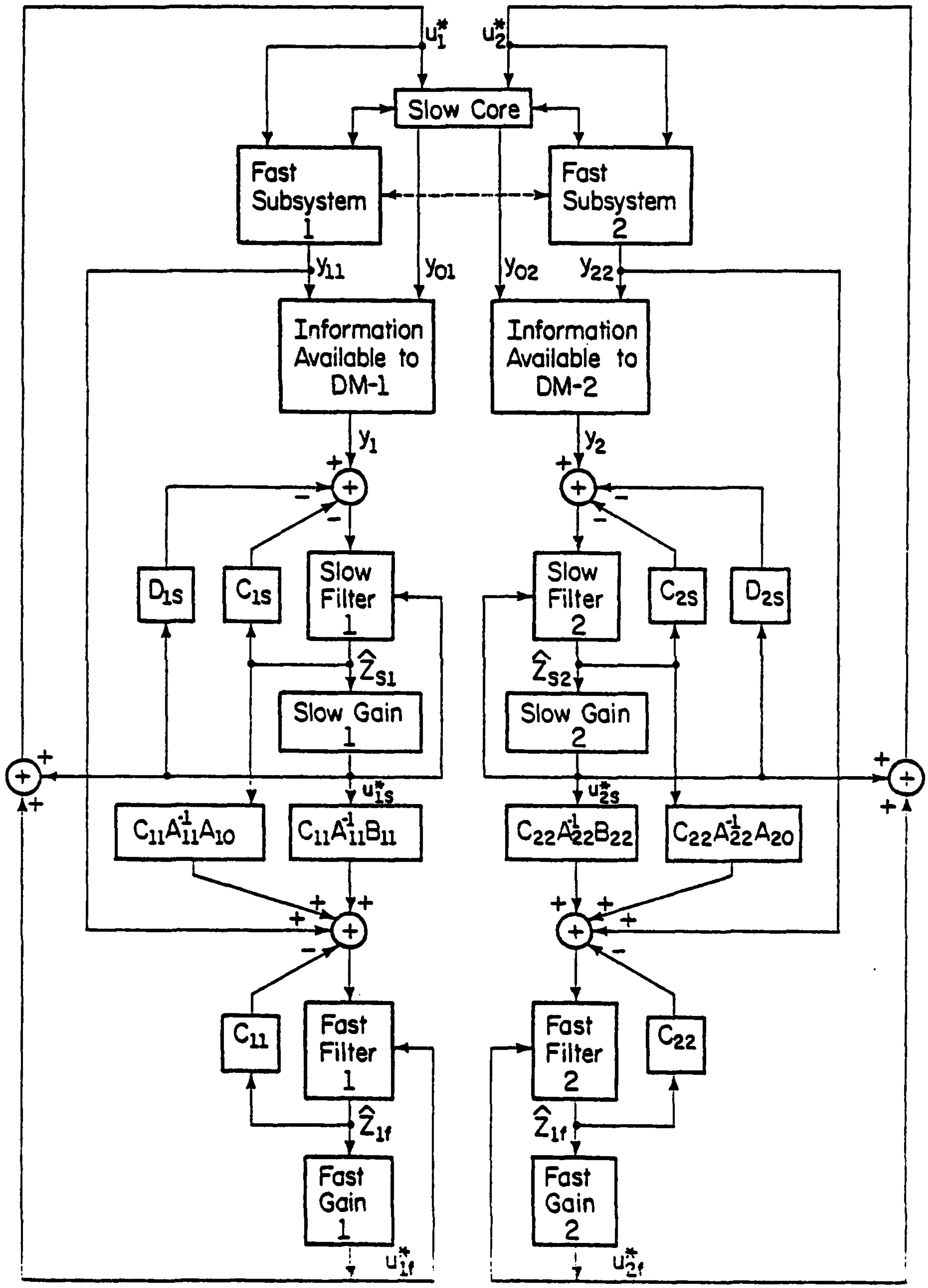

Fig. 3.1. Near-optimal filtering-control scheme.

CHAPTER 4

A MULTIMODEL APPROACH TO STOCHASTIC TEAM PROBLEMS

4.1. Introduction

In this chapter we continue to study the role of time-scales in multimodeling of stochastic linear systems. We shall demonstrate the well-posedness of multimodel generation by "k-th parameter perturbation" for both static and dynamic team problems under certain quasi-classical information structures. The weak-coupling assumption on the fast subsystems is retained.

In Section 4.2, the general dynamic team problem with sampled observations and quasi-classical information pattern is formulated. In Section 4.3, a multimodel solution is obtained for the static team problem. Then, in Section 4.4, the solution of the static team problem is utilized to obtain a multimodel solution to the dynamic team problem under the one-step-delay observation-sharing pattern. In both cases, the multimodel solution is shown to be well-posed; in the sense that it is the asymptotic limit of the optimal solution as the small parameters go to zero. The chapter concludes with Section 4.5.

4.2. Problem Formulation

The system under consideration consists of strongly-coupled slow core and weakly-coupled fast subsystems controlled by two decision makers. It is modeled by the Ito differential equations

$$dZ_0 = (A_{00}Z_0 + \sum_{j=1}^{2} (A_{0j}Z_j + B_{0j}u_j))\, dt + \sum_{j=1}^{2} F_{0j}dw_j;\ Z_0(t_0) = Z_{00}$$

$$\varepsilon_i dZ_i = (A_{i0}Z_0 + A_{ii}Z_i + \varepsilon_{ii}A_{ik}Z_k + B_{ii}u_i)\, dt + \sqrt{\varepsilon_i}F_{ii}dw_i;\ Z_i(t_0) = Z_{i0}$$

$$t \geq t_0;\ i,\ k=1,2;\quad i \neq k \tag{4.1}$$

where $\dim Z_0 = n_0$, $\dim Z_i = n_i$, and $\{u_i(t);\ t \geq t_0\}$ are m_i-dimensional stochastic processes denoting the controls of DM_i. $\{w_i(t);\ t \geq t_0;\ i=1, 2\}$ are standard Wiener processes independent of each other. The small singular perturbation parameters $\varepsilon_i > 0$ represent small time-constants, inertias, masses, etc.; while the small regular perturbation parameters ε_{ii} represent weak-coupling between the subsystems. The states $\{Z_i;\ i=1,2\}$ are fast since their derivatives are of order $1/\varepsilon_i$. The matrices $(A_{ii};\ i=1,2)$ are assumed to be nonsingular.

The initial conditions are assumed to have Gaussian statistics with known parameters which will be specified later. The decision makers make independent decentralized sampled measurements. Specifically, it is assumed that a p_i-dimensional observation

$$y_i(j) = C_{i0}Z_0(t_j) + C_{ii}Z_i(t_j) + v_i(j);\ i=1,2 \tag{4.2}$$

is available to DM_i at the sampled time instant t_j where $j = 0,1, \dots N-1$ and $t_0 < t_1 < \dots < t_{N-1} < t_N = t_f$. Denote the index set of time samples by $\theta = \{0,1, \dots, N-1\}$. Then the random vectors $\{v_i(j),\ j \in \theta,\ i=1,2\}$ are assumed to have independent Gaussian statistics $\{v_i(j) \sim N(0,R_{ij}),\ R_{ij} > 0,\ j \in \theta,\ i = 1,2\}$, and are independent of the process noise $w_i(t)$ and the initial conditions.

To exhibit the slow and fast variables explicitly, we use the following transformation:

$$\begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} I_0 - \varepsilon_1 M_1 N_1 - \varepsilon_2 M_2 N_2 & -\varepsilon_1 M_1 & -\varepsilon_2 M_2 \\ N_1 & I_1 & 0 \\ N_2 & 0 & I_2 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} \tag{4.3a}$$

which has an explicit inverse

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} I_0 & \varepsilon_1 M_1 & \varepsilon_2 M_2 \\ -N_1 & I_1 - \varepsilon_1 N_1 M_1 & -\varepsilon_2 N_1 M_2 \\ -N_2 & -\varepsilon_1 N_2 M_1 & I_2 - \varepsilon_2 N_2 M_2 \end{bmatrix} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \end{bmatrix} \tag{4.3b}$$

where $\{M_i, N_i;\ i=1,2\}$ satisfy

$$A_{ii}N_i - A_{i0} - \varepsilon_i N_i\ (A_{00} - A_{0i}N_i - A_{0k}N_k) + \varepsilon_{ii}A_{ik}N_k = 0$$

$$M_i(A_{ii} + \varepsilon_i N_i A_{0i}) - A_{0i} + \varepsilon_k M_k N_k A_{0i} - \varepsilon_i(A_{00} - A_{0i}N_i - A_{0k}N_k)M_i + \varepsilon_{kk}M_k A_{ki} = 0$$

$$i,k = 1,2;\ i \neq k\ . \tag{4.4}$$

The existence of solutions to (4.4) is guaranteed by the assumption that $(A_{ii};\ i=1,2)$ are nonsingular [15].

The transformed system and the observations of each DM can now be written down as,

$$d\eta_0 = (A_0(\varepsilon)\eta_0 + \sum_{j=1}^{2} \hat{B}_{0j}(\varepsilon)u_j)\ dt + \sum_{j=1}^{2} \hat{F}_{0j}(\varepsilon)\ dw_j;\ \eta_0(t_0) = \eta_{00}$$

$$\varepsilon_i d\eta_i = (A_i(\varepsilon)\eta_i + \hat{A}_{ik}(\varepsilon)\eta_k + \hat{B}_{ii}(\varepsilon)u_i + \hat{B}_{ik}(\varepsilon)u_k)\ dt + \sqrt{\varepsilon_i}(\hat{F}_{ii}(\varepsilon)dw_i$$

$$+ \hat{F}_{ik}(\varepsilon)dw_k); \qquad \eta_i(t_0) = \eta_{i0} \tag{4.5a}$$

$$y_i(j) = \hat{C}_{i0}(\varepsilon)\eta_0(t_j) + \hat{C}_{ii}(\varepsilon)\eta_i(t_j) + \hat{C}_{ik}(\varepsilon)\eta_k(t_j) + v_i(j)$$

$$t \geq t_0;\ i,k = 1,2;\ i \neq k;\ j \in \theta \tag{4.5b}$$

where

$$A_0(\varepsilon) = A_{00} - A_{01}N_1 - A_{02}N_2$$

$$A_i(\varepsilon) = A_{ii} + \varepsilon_i N_i A_{0i}$$

$$\hat{A}_{ik}(\varepsilon) = \varepsilon_i N_i A_{0k} + \varepsilon_{ii} A_{ik}$$

$$\hat{B}_{0i}(\varepsilon) = B_{0i} - M_i B_{ii} - \varepsilon_i M_i N_i B_{0i} - \varepsilon_k M_k N_k B_{0i}$$

$$\hat{B}_{ii}(\varepsilon) = B_{ii} + \varepsilon_i N_i B_{0i}$$

$$\hat{B}_{ik}(\varepsilon) = \varepsilon_i N_i B_{0k}$$

$$\hat{C}_{i0}(\varepsilon) = C_{i0} - C_{ii} N_i$$

$$\hat{C}_{ii}(\varepsilon) = C_{ii} - \varepsilon_i C_{ii} N_i M_i + \varepsilon_i C_{i0} M_i$$

$$\hat{C}_{ik}(\varepsilon) = \varepsilon_k C_{i0} M_k - \varepsilon_k C_{ii} N_i M_k$$

$$\hat{F}_{0i}(\varepsilon) = F_{0i} - \sqrt{\varepsilon_i} M_i F_{ii} - \varepsilon_i M_i N_i F_{0i} - \varepsilon_k M_k N_k F_{0i}$$

$$\hat{F}_{ii}(\varepsilon) = F_{ii} + \sqrt{\varepsilon_i} N_i F_{0i}$$

$$\hat{F}_{ik}(\varepsilon) = \sqrt{\varepsilon_i} N_i F_{0k}\ ;\ i,\ k=1,2;\ i \neq k. \tag{4.6}$$

Notice that in this representation the slow and fast dynamics are completely decoupled and further as $\|\varepsilon\|\to 0$ ($\varepsilon=[\varepsilon_1\varepsilon_2\varepsilon_{11}\varepsilon_{22}]$), the system matrix of (4.5) becomes block-diagonal. Without loss of generality, we shall be working with the representation (4.5) instead of (4.1) and (4.2).

With respect to the representation (4.5), assume that the statistics of the initial state vector are given by

$$\begin{bmatrix} \eta_{00} \\ \eta_{10} \\ \eta_{20} \end{bmatrix} \sim N\left[\bar{x}_0 = \begin{pmatrix} \bar{\eta}_{00} \\ \bar{\eta}_{10} \\ \bar{\eta}_{20} \end{pmatrix}, \Sigma_0 = \begin{pmatrix} \Sigma_{00} & \sqrt{\varepsilon_1}\Sigma_{01} & \sqrt{\varepsilon_2}\Sigma_{02} \\ \sqrt{\varepsilon_1}\Sigma'_{01} & \Sigma_{11} & \sqrt{\varepsilon_1\varepsilon_2}\,\Sigma_{12} \\ \sqrt{\varepsilon_2}\Sigma'_{02} & \sqrt{\varepsilon_1\varepsilon_2}\,\Sigma'_{12} & \Sigma_{22} \end{pmatrix}\right]. \tag{4.7}$$

The reason for assuming the particular form of the covariance matrix Σ_0 in (4.7) is because, together with (4.5a), we get $\mathrm{cov}(\eta_0,\eta_i) = 0(\sqrt{\varepsilon_i})$ and $\mathrm{cov}\,(\eta_1,\eta_2) = 0(\sqrt{\varepsilon_1\varepsilon_2})$ for all $t \geq t_0$. If we drop the small parameters from Σ_0, then the above covariance relations will hold only for $t > t_0$ outside some boundary-layer. The results obtained in the sequel would still be true since the contribution of the boundary-layer terms is only $0(\|\varepsilon\|)$. Assuming the particular form in (4.7) simplifies the algebra.

We now adopt a quasi-classical information pattern for this decision problem, and follow the formulation of [36]. Specifically, it is assumed that the DMs exchange their independent sampled observations with a delay of one sampling interval. Such an information pattern is known as the one-step-delay observation-sharing pattern [37]. Hence, the information available to DM_i in the time interval

$$[t_j,t_{j+1}] \text{ is } \alpha_i^j \text{ where } \alpha_i^j = \{y_i(j),\zeta_{j-1}\} \tag{4.8a}$$

and ζ_{j-1} denotes the common information available to the decision makers in the same sampling interval, i.e.;

$$\zeta_{j-1} = \{y_1(j-1), y_2(j-1), \ldots, y_1(0), y_2(0)\}. \tag{4.8b}$$

Let σ_i^j denote the sigma-algebra generated by the information set α_i^j. Further, let H_i^N denote the class of second-order stochastic processes $\{u_i(t), t \geq t_0\}$ which satisfy the requirement that their restriction to the interval $[t_j, t_{j+1})$ is σ_i^j-measurable, for all $j \in \theta$. Then a permissible strategy for DM_i is a mapping $\nu_i: [t_0, t_f] \times \mathbb{R}^{(p_1+p_2)N} \to \mathbb{R}^{mi}$, such that $\nu_i(\cdot, \alpha_i) \in H_i^N$. Denote the class of all such strategies for DM_i by Γ_i^N. It should be noted that for each pair of elements in $H_1^N \times H_2^N$, the stochastic differential equation (4.3a) admits a unique solution whose sample paths are continuous [38].

For each $(\nu_1 \in \Gamma_1^N, \nu_2 \in \Gamma_2^N)$, we now define the quadratic, strictly convex cost function as

$$J(\nu_1, \nu_2) = E\{\eta_0'(t_f)Q_{0f}\eta_0(t_f) + \sum_{i=1}^{2} \varepsilon_i \eta_i'(t_f)Q_{if}\eta_i(t_f) + \int_{t_0}^{t_f} (\eta_0'Q_0\eta_0$$
$$+ \sum_{i=1}^{2} (\eta_i'Q_i\eta_i + u_i'u_i))\, dt \,|\, u_i(t) = \nu_i(t, \alpha_i),\ i=1,2\} \tag{4.9}$$

where $\{Q_{if}, Q_i \geq 0;\ i=0,1,2\}$, and the expectation operator is taken over the underlying statistics.

Then an optimal solution for this dynamic team problem is a pair $\{\nu_i^* \in \Gamma_i,\ i=1,2\}$ such that

$$\inf_{\Gamma_1^N} \inf_{\Gamma_2^N} J(\nu_1, \nu_2) = J(\nu_1^*, \nu_2^*). \tag{4.10}$$

Defining $x' = [\eta_0'\eta_1'\eta_2']$ and $w' = [w_1'w_2']$, equations (4.5) and (4.9) can be written in a composite form as

$$dx = (Ax + \sum_{j=1}^{2} B_j u_j)dt + Fdw; \quad x(t_0) = x_0 \tag{4.11a}$$

$$y_i(j) = C_i\, x(t_j) + v_i(j); \ i=1,2; \ j \in \theta \tag{4.11b}$$

$$J(\nu_1,\nu_2) = E\{x'(t_f)Q_f x(t_f) + \int_{t_0}^{t_f} (x'Qx + u_1'u_1 + u_2'u_2)dt \,|\, u_i(t) = \nu_i(t,\alpha_i), \quad i=1,2\} \tag{4.12}$$

where

$$A = \begin{bmatrix} A_0(\varepsilon) & 0 & 0 \\ 0 & \frac{1}{\varepsilon_1} A_1(\varepsilon) & \frac{1}{\varepsilon_1} \hat{A}_{12}(\varepsilon) \\ 0 & \frac{1}{\varepsilon_2} \hat{A}_{21}(\varepsilon) & \frac{1}{\varepsilon_2} A_2(\varepsilon) \end{bmatrix}, \quad B_i = \begin{bmatrix} \hat{B}_{0i}(\varepsilon) \\ \frac{1}{\varepsilon_1} \hat{B}_{1i}(\varepsilon) \\ \frac{1}{\varepsilon_2} \hat{B}_{2i}(\varepsilon) \end{bmatrix}, \quad F = \begin{bmatrix} \hat{F}_{01}(\varepsilon) & \hat{F}_{02}(\varepsilon) \\ \frac{1}{\sqrt{\varepsilon_1}}\hat{F}_{11}(\varepsilon) & \frac{1}{\sqrt{\varepsilon_1}}\hat{F}_{12}(\varepsilon) \\ \frac{1}{\sqrt{\varepsilon_2}}\hat{F}_{21}(\varepsilon) & \frac{1}{\sqrt{\varepsilon_2}}\hat{F}_{22}(\varepsilon) \end{bmatrix}$$

$$C_i = [\hat{C}_{i0}(\varepsilon)\ \hat{C}_{i1}(\varepsilon)\ \hat{C}_{i2}(\varepsilon)]$$

$$Q_f = \text{block diag } [Q_{0f}, \varepsilon_1 Q_{1f}, \varepsilon_2 Q_{2f}]$$

$$Q = \text{block diag } [Q_0, Q_1, Q_2]. \tag{4.13}$$

The following assumptions are made in order to guarantee the existence of a unique limit, as $\|\varepsilon\| \to 0$, of the optimal solution.

Assumptions:

a) $\text{Re}\ \lambda(A_{ii}) < 0; \quad i=1,2$

b) $(A_{ii}, B_{ii}, \sqrt{Q_i})$ is controllable-observable; $i=1,2$.

Before obtaining the solution of the dynamic team problem defined by (4.10), (4.11), and (4.12), we first consider its static version (obtained by setting N=1) in the next section.

4.3. Static Team Problem

In the static version of the dynamic team problem formulated in the last section, the decision makers make noisy linear observations of the random initial state, and do not require any further information as the decision process proceeds. Hence, the static version can be recovered from the general formulation by setting N=1.

To this end, let the observation y_i of DM_i be given as

$$y_i = C_i x_0 + v_i; \; i=1,2 \tag{4.14}$$

where $v_i \sim N(0,R_i)$ and $x_0 \sim N(\bar{x}_0,\Sigma_0)$, and these random vectors are statistically independent.

An optimal solution for the static team problem defined by (4.11a), (4.14), and (4.12) is a pair $\{\nu_i^* \in \Gamma_i^1, \; i=1,2\}$ such that

$$\inf_{\Gamma_1^1} \inf_{\Gamma_2^1} J(\nu_1,\nu_2) = J(\nu_1^*,\nu_2^*) \,. \tag{4.15}$$

The unique optimal solution to this problem is given in [36], and can also be found in Appendix C.

Due to the presence of widely separated eigenvalues, the differential equations (C2)-(C17) involved for computing the optimal solution are numerically stiff. This renders the optimal solution computationally infeasible, specially when the order of the system is very large. Sometimes it is even difficult to obtain the optimal solution; e.g., when the small perturbation parameters are unknown, or when one DM does not have a knowledge of the fast dynamics of the other DM. In such cases we need to look for other suboptimal solutions. The multimodel solution proposed here does not require every DM to have an exact knowledge of the fast dynamics of other DMs.

Moreover, as we shall see later, it is well-posed in the sense that it tends to the optimal solution in the limit as the small parameters go to zero.

Before we propose the multimodel solution to the static team problem, we need the following result from Chapter 3.

Lemma 4.1: Let $\eta_i(t)$ satisfy the Ito differential equation

$$\varepsilon_i d\eta_i = (A_{ii}\eta_i + B_{ii}u_i)\, dt + \sqrt{\varepsilon_i} F_{ii} dw_i \tag{4.16}$$

where w_i is a standard Wiener process, u_i is a known function of time, and $\mathrm{Re}\lambda(A_{ii}) < 0$. Then $\eta_i(t) \to \eta_{is}(t)$ weakly as $\varepsilon_i \to 0$, where

$$\eta_{is}(t) = -A_{ii}^{-1}\, B_{ii}\, u_i(t) + \tilde{w}_i\,, \tag{4.17}$$

and $\tilde{w}_i$ is a constant zero mean Gaussian random vector with variance $\tilde{W}_i$ satisfying the Lyapunov equation

$$A_{ii}\, \tilde{W}_i + \tilde{W}_i\, A'_{ii} + F_{ii}\, F'_{ii} = 0. \tag{4.18}$$

The weak limit of $\eta_i(t)$ has been shown to be the appropriate limit for eliminating the variable $\eta_i(t)$ from the cost functional J to obtain the slow cost (Chapter 3).

The multimodel solution is obtained by solving the following low-order problems.

4.3.1. Slow subproblem

This is a static team problem obtained by taking the limit as $\|\varepsilon\| \to 0$ in the original problem defined by (4.11a), (4.14), and (4.12).

$$d\eta_{0s} = (A_0\eta_{0s} + \sum_{i=1}^{2} B_{0i}\, u_{is})\, dt + \sum_{i=1}^{2} F_{0i}\, dw_i;\ \eta_{0s}(t_0) = \eta_{00} \tag{4.19}$$

$$y_{is} = \hat{C}_{i0}\, \eta_{00} + v_i \equiv y_i - C_{ii}\, \eta_{i0};\ i=1,2 \tag{4.20}$$

$$\eta_{00} \sim N(\bar{\eta}_{00}, \Sigma_{00});\ v_i \sim N(0, R_i) \tag{4.21}$$

$$J_s(\nu_{1s}, \nu_{2s}) = E\,\{\eta_{0s}'(t_f)\, Q_{0f}\, \eta_{0s}(t_f) + \int_{t_0}^{t_f} (\eta_{0s}'\, Q_0\, \eta_{0s} + \sum_{i=1}^{2} u_{is}'\, R_{is}\, u_{is})$$

$$dt \,|\, u_{is}(t) = \nu_{is}(t, \alpha_i),\ i=1,2\} + J_0 \tag{4.22a}$$

where

$$R_{is} = I + (A_{ii}^{-1}\, B_{ii})'\, Q_i\, (A_{ii}^{-1}\, B_{ii}) \tag{4.22b}$$

$$J_0 = (t_f - t_0) \sum_{i=1}^{2} \mathrm{tr}\,(Q_i \tilde{W}_i) \tag{4.22c}$$

$\tilde{W}_i$ is the symmetric nonnegative definite solution of the Lyapunov equation (4.18).

The unique optimal team solution to the slow subproblem defined by (4.19)-(4.22) is given by Theorem 2 of [36]:

$$u_{is}^*(t) = P_{is}\, [y_i - \hat{C}_{i0}\bar{\eta}_{00} - C_{ii}\bar{\eta}_{i0}] - R_{is}^{-1}\, B_{0i}'\, S_s\, \bar{\eta}_{0s}(t);\ i=1,2 \tag{4.23}$$

where $S_s(t)$ is the nonnegative definite solution of the Riccati equation

$$\dot{S}_s + A_0' S_s + S_s A_0 - S_s\, (E_{1s} + E_{2s})\, S_s + Q_0 = 0;\ S_s(t_f) = Q_{0f} \tag{4.24a}$$

$$\dot{\bar{\eta}}_{0s}(t) = (A_0 - E_{1s} S_s - E_{2s} S_s)\, \bar{\eta}_{0s}(t);\ \bar{\eta}_{0s}(t_0) = \bar{\eta}_{00} \tag{4.24b}$$

$$P_{is} = R_{is}^{-1}\, B_{0i}'\, S_{is}\, [\tilde{P}_{is} - \tilde{L}_{js}\Sigma_{is}] - R_{is}^{-1}\, B_{0i}'\, K_{is};\ i,j=1,2;\ i \neq j \tag{4.24c}$$

$S_{is}(t)$ is nonnegative definite solution of the Riccati equation

$$\dot{S}_{is} + A_0' S_{is} + S_{is} A_0 - S_{is} E_{is} S_{is} + Q_0 = 0;\ S_{is}(t_f) = Q_{0f};\ i=1,2, \tag{4.24d}$$

and

$$\dot{\tilde{P}}_{is} = [A_0 - E_{is} S_{is}]\, \tilde{P}_{is} + E_{is}\, [K_{is} + S_{is} \tilde{L}_{js} \Sigma_{is}];\ P_{is}(t_0) = 0;\ i,j=1,2;\ i \neq j \tag{4.24e}$$

$$\dot{\tilde{L}}_{is} = A_0\, \tilde{L}_{is} + E_{is}\, S_{is}\, [\tilde{P}_{is} - \tilde{L}_{js} \Sigma_{is}]\, \hat{C}_{i0} - E_{is}\, K_{is}\, \hat{C}_{i0};\ \tilde{L}_{is}(t_0) = I;$$

$$i,j=1,2;\ i \neq j \tag{4.24f}$$

$$\dot{K}_{is} = -[A_0 - E_{is} S_{is}]'\, K_{is} - S_{is}\, E_{js}\, S_{js}\, [\tilde{P}_{js} - N_{is} \Sigma_{js}]\, \hat{C}_{j0}\, \Sigma_{is}$$

$$+ S_{is} E_{js} K_{js} \hat{C}_{j0} \Sigma_{is};\quad K_{is}(t_f) = 0;\quad i,j = 1,2;\quad i \neq j \tag{4.24g}$$

$$E_{is} = B_{0i}\, R_{is}^{-1}\, B_{0i}';\ \Sigma_{is} = \Sigma_{00}\, \hat{C}_{i0}'\, [\hat{C}_{i0} \Sigma_{00} \hat{C}_{i0}' + C_{ii} \Sigma_{ii} C_{ii}' + R_i]^{-1};\ i=1,2. \tag{4.24h}$$

The minimum value of J_s^* is given by

$$J_s^* = J_s(u_{1s}^*, u_{2s}^*) = \bar{\eta}_{00}'\, S_s(0)\, \bar{\eta}_{00} + \mathrm{tr}\, (\Sigma_{00} S_s(0))$$

$$+ \mathrm{tr}(\int_{t_0}^{t_f} S_s(t) \sum_{i=1}^{2} F_{0i} F_{0i}' dt) + J_{ms} \tag{4.25a}$$

where

$$J_{ms} = \mathrm{tr} \int_{t_0}^{t_f} [\sum_{i=1}^{2} (\Lambda_{0s}^{(i)'}\, \Lambda_{0s}^{(i)}\, \Sigma_{00} + \Lambda_{is}^{(1)'}\, \Lambda_{is}^{(1)}\, R_i + \Lambda_{is}^{(2)'}\, \Lambda_{is}^{(2)}\, R_i$$

$$+ S_s\, E_{is}\, S_s\, W_s)]\, dt \tag{4.25b}$$

with

$$\Lambda_{0s}^{(i)}(t) = P_{is}\, \hat{C}_{i0} - R_{is}^{-1}\, B_{0i}'\, S_s\, (\tilde{L}_{is} + \tilde{L}_{js}) + 3R_{is}^{-1}\, B_{0i}'\, S_s\, \Phi_0\, (t,t_0);$$

$$i,j=1,2;\ i \neq j \tag{4.26a}$$

$$\Lambda_{is}^{(i)}(t) = P_{is} + R_{is}^{-1}\, B_{0i}'\, S_s\, V_{is};\ i=1,2 \tag{4.26b}$$

$$\Lambda_{is}^{(j)}(t) = R_{js}^{-1}\, B_{0j}'\, S_s\, V_{is};\ i,j=1,2;\ i \neq j \tag{4.26c}$$

$$\dot{V}_{is} = A_0\, V_{is} + E_{is}\, S_{is}\, [\tilde{P}_{is} - \tilde{L}_{js}\Sigma_{is}] - E_{is}\, K_{is};\ V_{is}(t_0) = 0;$$

$$i,j=1,2;\ i \neq j \tag{4.26d}$$

$$\dot{W}_s = A_0\, W_s + W_s\, A_0' + F_{01}\, F_{01}' + F_{02}\, F_{02}' = 0\ ;\ W_s(t_0) = 0 \tag{4.26e}$$

$$\dot{\Phi}_0 = A_0\, \Phi_0;\ \Phi_0(t_0,t_0) = I\,. \tag{4.26f}$$

4.3.2. Fast subproblems

These are stochastic control problems for each DM (i=1,2), which are defined by the state equation

$$\varepsilon_i d\eta_{if} = (A_{ii}\eta_{if} + B_{ii}\, u_{if})\, dt + \sqrt{\varepsilon_i}\, F_{ii}\, dw_i;\ \eta_{if}(t_0) = \eta_{i0} \tag{4.27}$$

the initial state measurement

$$y_{if} = C_{ii}\eta_{i0} + v_i \equiv y_i - \hat{C}_{i0}\, \eta_{00} \tag{4.28a}$$

$$\eta_{i0} \sim N(\bar{\eta}_{i0},\, \Sigma_{ii});\ v_i \sim N(0,R_i) \tag{4.28b}$$

and the cost functional

$$J_{if}(\nu_{if}) = E\{\varepsilon_i \eta_{if}'(t_f)\, Q_{if} \eta_{if}(t_f) + \int_{t_0}^{t_f} (\eta_{if}' Q_{if} \eta_{if} + u_{if}' u_{if}) dt \,|\, u_{if}(t) = \nu_{if}(t,\alpha_i)\}. \tag{4.29}$$

The unique optimal solution to these static control problems is given by

$$u_{if}^*(t) = -B_{ii}' S_{if} \Psi_{if}(t,t_0) \Sigma_{if} \{y_i - \hat{C}_{i0}\bar{\eta}_{00} - C_{ii}\bar{\eta}_{i0}\} - B_{ii}' S_{if} \bar{\eta}_{if}(t), \tag{4.30}$$

where $S_{if}(t)$ is the nonnegative definite solution of the Riccati equation

$$\varepsilon_i \dot{S}_{if} = -A_{ii}' S_{if} - S_{if} A_{ii} - Q_i + S_{if} B_{ii} B_{ii}' S_{if};\; S_{if}(t_f) = Q_{if}, \tag{4.31a}$$

$\Psi_{if}(t,t_0)$ is the state transition matrix of the system

$$\varepsilon_i \dot{\bar{\eta}}_{if}(t) = (A_{ii} - B_{ii} B_{ii}' S_{if})\, \bar{\eta}_{if}(t);\; \bar{\eta}_{if}(t_0) = \bar{\eta}_{i0}, \tag{4.31b}$$

and

$$\Sigma_{if} = \Sigma_{ii} C_{ii}' (\hat{C}_{i0} \Sigma_{00} \hat{C}_{i0}' + C_{ii} \Sigma_{ii} C_{ii}' + R_i)^{-1}. \tag{4.31c}$$

The minimum value of J_{if}^* is given by

$$J_{if}^* = J_{if}(u_{if}^*) = \varepsilon_i \bar{\eta}_{i0}' S_{if}(0) \bar{\eta}_{i0} + \varepsilon_i \mathrm{tr}(\Sigma_{ii} S_{if}(0)) + \mathrm{tr}(\int_{t_0}^{t_f} S_{if} F_{ii} F_{ii}' dt) + J_{mf}^i \tag{4.32a}$$

where

$$J_{mf}^i = \mathrm{tr}\{\int_{t_0}^{t_f} (\Lambda_{0f}^{(i)'} \Lambda_{0f}^{(i)} \Sigma_{ii} + \Lambda_{if}^{(i)'} \Lambda_{if}^{(i)} R_i + S_{if} B_{ii} B_{ii}' S_{if} \tilde{W}_i) dt\} \tag{4.32b}$$

with

$$A_{0f}^{(i)}(t) = 2(B_{ii}' S_{if} \Phi_i (t,t_0) - B_{ii}' S_{if} \Psi_{if} (t,t_0) \Sigma_{if} C_{ii}) \quad (4.33a)$$

$$A_{if}^{(i)}(t) = -B_{ii}' S_{if} \Phi_i (t,t_0) \Sigma_{if} \quad (4.33b)$$

$$\varepsilon_i \dot{\Phi}_i (t,t_0) = A_{ii} \Phi_i (t,t_0); \; \Phi_i (t_0,t_0) = I. \quad (4.33c)$$

Under Assumption b, $S_{if}(t) \to \bar{S}_{if}$ as $\varepsilon_i \to 0$, where $\bar{S}_{if}$ is the unique positive definite solution of the algebraic Riccati equation

$$A_{ii}' \bar{S}_{if} + \bar{S}_{if} A_{ii} + Q_i - \bar{S}_{if} B_{ii} B_{ii}' \bar{S}_{if} = 0 . \quad (4.34)$$

Also, $J_{if}^* \to \bar{J}_{if}^*$ as $\varepsilon_i \to 0$, where

$$\bar{J}_{if}^* = tr \left(\int_{t_0}^{t_f} S_{if} F_{ii} F_{ii}' dt\right) + \bar{J}_{mf}^i \quad (4.35)$$

and $\bar{J}_{mf}^i$ is J_{mf}^i with S_{if} replaced by $\bar{S}_{if}$.

The optimal control $u_{if}^*(t)$ tends to $\bar{u}_{if}^*(t)$ as $\varepsilon_i \to 0$, where $\bar{u}_{if}^*(t)$ is $u_{if}^*(t)$ with S_{if} replaced by $\bar{S}_{if}$.

The multimodel strategy pair $\{u_{im}(t);\ i=1,2\}$ is formed by combining the optimal strategies of the slow and fast subproblems.

$$\begin{aligned} u_{im}(t) &= u_{is}^*(t) + \bar{u}_{if}^*(t) \\ &= [P_{is}(t) - B_{ii}' \bar{S}_{if} \Psi_{if}(t,t_0) \Sigma_{if}] \, [y_i - \hat{C}_{i0} \bar{\eta}_{00} - C_{ii} \bar{\eta}_{i0}] \\ &\quad - R_{is}^{-1} B_{0i}' S_s \bar{\eta}_{0s}(t) - B_{ii}' \bar{S}_{if} \bar{\eta}_{if}(t) \end{aligned} \quad (4.36)$$

The following proposition now establishes the well-posedness of the multimodel solution.

<u>Proposition 4.1</u>:

a) $u_i^*(t) = u_{im}(t) + 0(\|\varepsilon\|)$; $i=1,2$; $t \in [t_1,t_2] \subset [t_0,t_f]$

b) $J^* \quad = J_s^* + \bar{J}_{1f}^* + \bar{J}_{2f}^* + 0(\|\varepsilon\|)$.

<u>Proof</u>: See Appendix C.

4.4. Dynamic Team Problem

We now obtain the solution of the dynamic team problem formulated in Section 4.2. This is done by first enlarging the strategy spaces of the DMs so as to formulate a new team problem whose optimal solution can be obtained more readily. The solution of the original problem is then obtained from the solution to the new problem.

The new team problem differs from the old one in the information patterns of the DMs. Specifically, the new one is defined by replacing α_i^j and ζ_{j-1} given by (4.8), by $\tilde{\alpha}_i^j$ and $\tilde{\zeta}_{j-1}$, respectively, where

$$\tilde{\alpha}_i^j \quad = \{y_i(j),\ \tilde{\zeta}_{j-1}\} \tag{4.37a}$$

$$\tilde{\zeta}_{j-1} = \{\zeta_{j-1};\ u_1(t),\ u_2(t),\ t < t_j\} \quad . \tag{4.37b}$$

Under this new information pattern, the DMs also have access to each other's control values used during all past sampling intervals. This information pattern, though not of much practical importance, is mathematically convenient for obtaining the solution to the original problem due to the following fact [36]:

$$\min_{\Gamma_1^N} \min_{\Gamma_2^N} J(\nu_1,\nu_2) = \min_{\tilde{\Gamma}_1^N} \min_{\tilde{\Gamma}_2^N} J(\nu_1,\nu_2) \tag{4.38}$$

where $\tilde{\Gamma}_1^N$ and $\tilde{\Gamma}_2^N$ are defined analogous to Γ_1^N and Γ_2^N respectively, but under the new information pattern.

For each $\{\tilde{\nu}_1 \in \tilde{\Gamma}_1^N, \tilde{\nu}_2 \in \tilde{\Gamma}_2^N\}$, the implicit equations

$$\tilde{\nu}_i^j\ (\tilde{\alpha}_i^j) = u_i^j(\omega);\ i=1,2;\ j = N-1, \ldots, 0. \tag{4.39}$$

can be solved recursively for $\{u_i^j(\omega),\ j = N-1, \ldots, 0;\ i=1,2\}$ as functions of $\{\alpha_i^j,\ j=N-1, \ldots, 0;\ i=1,2\}$ because of the nature of the information pattern. Then the resulting functional relations provide a pair in $\Gamma_1^N \times \Gamma_2^N$, and a unique one since the stochastic differential equation (4.5a) admits a unique solution in each sampling interval. In fact, there exist uncountably many pairs in $\tilde{\Gamma}_1^N \times \tilde{\Gamma}_2^N$ corresponding to a given pair in $\Gamma_1^N \times \Gamma_2^N$; equivalently, a pair of strategies under the original information structure has several representations under the new (enlarged) information pattern [10]. In [36] one such representation in $\tilde{\Gamma}_1^N \times \tilde{\Gamma}_2^N$ is first obtained which is the simplest to derive. Then implicit equations of the type (4.39) are solved to obtain the desired optimal team solution.

The optimal solution to the dynamic team problem involves solving an appropriate static team problem with respect to the current outputs, within each sampling interval. The shared information affects the statistics of the initial state at the beginning of each sampling interval. The computational problem worsens, since now we need to solve a set of stiff differential equations in every sample interval. Hence, a suboptimal solution without such numerical stiffness will be much more desirable in the dynamic case.

The multimodel solution, which is one such suboptimal solution, is obtained by solving the following low-order problems.

4.4.1. Slow subproblem

This is a dynamic team problem obtained by taking the limit as $\|\varepsilon\| \to 0$ in the original problem defined by (4.11)-(4.13).

The state equation for this problem is given by (4.19), the cost criterion by (4.22), and observations by

$$y_{is}(j) = \hat{C}_{i0}\eta_{0s}(t_j) + v_i(j) \equiv y_i(j) - C_{ii}\eta_{is}(t_j)$$

$$i=1,2;\ j \in \theta\ . \tag{4.40}$$

The optimal solution to the slow dynamic team problem under the new information structure (4.37) is given by [36],

$$\tilde{v}^*_{is}(t,\tilde{\alpha}_i) = \hat{P}_{is}(t)\,[y_i(j)-\hat{C}_{i0}\hat{\eta}_{0s}(t_j^-) - C_{ii}\hat{\eta}_{is}(t_j^-)]$$

$$-R_{is}^{-1}B_{0i}'S_s\Psi_{0s}(t,t_j)\hat{\eta}_{0s}(t_j^-)\ ;\quad i=1,2;\quad t\in[t_j,t_{j+1}),\ j\in\theta. \tag{4.41}$$

where

$$\dot{\Psi}_{0s}(t,t_j) = (A-E_{1s}S_s-E_{2s}S_s)\ \Psi_{0s}(t,t_j);\ t\in [t_j,t_{j+1});\ \Psi_{0s}(t_j,t_j) = I. \quad (4.42a)$$

$$\hat{P}_{is}(t) = R_{is}^{-1}\ B_{0i}'\ \bar{S}_{is}\ [\bar{P}_{is}-\bar{L}_{ks}\bar{\Sigma}_{is}(j)] - R_{is}^{-1}\ B_{0i}'\ \bar{K}_{is};\ t\in [t_j,t_{j+1})\ ;$$

$$i,k=1,2;\ i\neq k;\ j\in\theta. \quad (4.42b)$$

$$\dot{\bar{P}}_{is} = [A_0-E_{is}\bar{S}_{is}]\ \bar{P}_{is} + E_{is}\ [\bar{K}_{is}+\bar{S}_{is}\bar{L}_{ks}\bar{\Sigma}_{is}(j)];\ t\in [t_j,t_{j+1});\ \bar{P}_{is}(t_j)=0\ ;$$

$$i,k=1,2;\ i\neq k;\ j\in\theta. \quad (4.42c)$$

$$\dot{\bar{L}}_{is} = A_0\bar{L}_{is}+E_{is}\bar{S}_{is}\ [\bar{P}_{is}-\bar{L}_{ks}\bar{\Sigma}_{is}(j)]\ \hat{C}_{i0}-E_{is}\bar{K}_{is}\hat{C}_{i0};\ t\in [t_j,t_{j+1});\ \bar{L}_{is}(t_j)=I\ ;$$

$$i,k=1,2;\ i\neq k;\ j\in\theta. \quad (4.42d)$$

$$\dot{\bar{K}}_{is} = -(A_0-E_{is}\bar{S}_{is})'\ \bar{K}_{is}-\bar{S}_{is}E_{ks}\bar{S}_{ks}\ [\bar{P}_{ks}-\bar{L}_{is}\bar{\Sigma}_{ks}(j)]\ \hat{C}_{k0}\bar{\Sigma}_{is}(j) + \bar{S}_{is}E_{ks}\bar{K}_{ks}\hat{C}_{k0}\bar{\Sigma}_{is}(j)$$

$$t\in [t_j,t_{j+1});\ \bar{K}_{is}(t_{j+1}) = 0;\ i,k=1,2;\ i\neq k;\ j\in\theta. \quad (4.42e)$$

$$\dot{\bar{S}}_{is} = -A_0'\bar{S}_{is}-\bar{S}_{is}A_0-Q_0 + \bar{S}_{is}\ E_{is}\ \bar{S}_{is};\ t\in [t_j,t_{j+1});\ \bar{S}_{is}(t_{j+1}) = S_s(t_{j+1})\ ;$$

$$i=1,2;\ j\in\theta. \quad (4.42f)$$

$$\bar{\Sigma}_{is}(j) = \Sigma_{00}(t_j^-)\ \hat{C}_{i0}'\ [\hat{C}_{i0}\bar{\Sigma}_{00}(t_j^-)\hat{C}_{i0}' + C_{ii}\ \tilde{W}_i\ C_{ii}' + R_{ij}]^{-1}\ ;$$

$$i=1,2;\ j\in\theta. \quad (4.42g)$$

$$\hat{\eta}_{0s}(t_j^-) = E[\eta_{0s}(t_i)|\tilde{\zeta}_{j-1}];\ j\in\theta. \quad (4.43a)$$

$$\mathrm{Cov}\ (\hat{\eta}_{0s}(t_j^-),\ \hat{\eta}_{0s}(t_j^-)) = \Sigma_{00}(t_j^-);\ j\in\theta. \quad (4.43b)$$

$$\left.\begin{aligned}
&\dot{\hat{\eta}}_{0s} = A_0\hat{\eta}_{0s} + B_1u_{1s} + B_2u_{2s};\ t \in [t_{j-1},t_j);\ j=1,2,\ \ldots,N;\\
&\hat{\eta}_{0s}(t_0) = \bar{\eta}_{00}\\
&\hat{\eta}_{0s}(t_j) = \hat{\eta}_{0s}(t_j^-) + \hat{K}_s(j)\ [y(j)-C_0\hat{\eta}_{0s}(t_j^-) - \sum_{i=1}^{2} \hat{C}_{ii}\ \hat{\eta}_{is}(t_j^-)]\\
&\hat{\eta}_{is}(t_j^-) = -A_{ii}^{-1}\ B_{ii}\ u_{is}(t_j^-);\ i=1,2.
\end{aligned}\right\} \quad (4.44)$$

$$\left.\begin{aligned}
&\dot{\bar{\Sigma}}_{00}(t) = A_0\bar{\Sigma}_{00} + \bar{\Sigma}_{00}A_0' + F_{01}F_{01}' + F_{02}F_{02}';\ t \in [t_{j-1},t_j);\\
&\bar{\Sigma}_{00}(t_0) = \Sigma_{00};\ j=1,2\ \ldots,N\\
&\bar{\Sigma}_{00}(t_j) = \bar{\Sigma}_{00}(t_j^-) - \hat{K}_s(j)\ C_0\ \bar{\Sigma}_{00}(t_j^-)
\end{aligned}\right\} \quad (4.45)$$

$$\left.\begin{aligned}
&\hat{K}_s(j) = \bar{\Sigma}_{00}(t_j^-)\ C_0'\ [C_0\ \bar{\Sigma}_{00}(t_j^-)\ C_0' + \sum_{i=1}^{2} \hat{C}_{ii}\ \tilde{W}_i\ \hat{C}_{ii}' + R_j]^{-1}\\
&C_0 = [\hat{C}_{10}'\ \hat{C}_{20}']'\\
&\hat{C}_{11} = [C_{11}'\ 0]'\\
&\hat{C}_{22} = [0\ C_{22}']'.
\end{aligned}\right\} \quad (4.46)$$

The unique optimal solution under the one-step-delay observation sharing pattern is given by

$$\nu_{is}^*(t,\alpha_i) = \hat{P}_{is}(t)\ [y_i(j)-\hat{C}_{i0}\hat{\eta}_{0s}^*(t_j^-)-C_{ii}\hat{\eta}_{is}^*(t_j^-)] - R_{is}^{-1}\ B_{0i}'\ S_s\ \Psi_{0s}(t,t_j)\hat{\eta}_{0s}^*(t_j^-)\ ;$$

$$i=1,2;\ t \in [t_j,t_{j+1});\ j \in \theta. \quad (4.47)$$

where $\hat{\eta}_{0s}^*(t)$, $\hat{\eta}_{is}^*(t)$ are the solutions of (4.44) with $u_{is}(t)$ replaced by $\nu_{is}^*(t,\alpha_i)$.

4.4.2. Fast subproblems

These are stochastic control problems with sampled observations for each DM (i=1,2). Each one is defined by the state equation (4.27), the cost criterion (4.29) and the observations

$$y_{if}(j) = C_{ii}\eta_{if}(t_j) + v_i(j) \equiv y_i(j) - \hat{C}_{i0}\eta_{0s}(t_j) - \hat{C}_{ii}\eta_{is}(t_j)\,; \quad j \in \theta\,. \tag{4.48}$$

The unique optimal solution to these control problems is given by

$$\begin{aligned}\tilde{u}^*_{if}(t) = -B'_{ii}\bar{S}_{if}\,\Psi_{if}(t,t_j)\,[\hat{\eta}^*_{if}(t_j^-) &+ \hat{K}_f(j)[y_i(j) - \hat{C}_{i0}\hat{\eta}^*_{0s}(t_j^-) - C_{ii}\hat{\eta}^*_{is}(t_j^-) \\ &- C_{ii}\hat{\eta}^*_{if}(t_j^-)]]\,;\; t \in [t_j, t_{j+1});\; j \in \theta.\end{aligned} \tag{4.49}$$

where $\bar{S}_{if}$ is the unique positive definite solution of (4.34), and

$$\varepsilon_i\,\dot{\Psi}_{if}(t,t_j) = (A_{ii} - B_{ii}B'_{ii}\bar{S}_{if})\,\Psi_{if}(t,t_j);\; t \in [t_j, t_{j+1});\; \Psi_{if}(t_j,t_j) = I. \tag{4.50}$$

$$\left.\begin{aligned}
\varepsilon_i\dot{\hat{\eta}}^*_{if} &= A_{ii}\hat{\eta}^*_{if} + B_{ii}\tilde{u}^*_{if};\; t \in [t_{j-1}, t_j);\; j \in 1,2,\ldots,N\,; \\
\hat{\eta}^*_{if}(t_0) &= \bar{\eta}_{i0}\,, \\
\hat{\eta}^*_{if}(t_j) &= \hat{\eta}^*_{if}(t_j^-) + \hat{K}_f(j)\,[y_i(j) - \hat{C}_{i0}\hat{\eta}^*_{0s}(t_j^-) - C_{ii}\hat{\eta}^*_{is}(t_j^-) - C_{ii}\hat{\eta}^*_{if}(t_j^-)]\,, \\
\hat{K}_f(j) &= \Sigma_{ii}C'_{ii}\,[\hat{C}_{i0}\Sigma_{00}\hat{C}'_{i0} + C_{ii}\Sigma_{ii}C'_{ii} + R_{ij}]^{-1};\; j=0\,, \\
&= \tilde{W}_i\,C'_{ii}\,[\hat{C}_{i0}\bar{\Sigma}_{00}(t_j^-)\hat{C}'_{i0} + C_{ii}\tilde{W}_iC'_{ii} + R_{ij}]^{-1};\; j=1,2,\ldots,N.
\end{aligned}\right\} \tag{4.51}$$

Remarks: Notice the sequential nature of the slow and fast subproblems. The parameters associated with the solution of the slow subproblem, namely, $\hat{\Pi}^*_{0s}(t_j)$, $\hat{\Pi}^*_{1s}(t_j)$, and $\bar{\Sigma}_{00}(t_j)$, enter the solution of the fast subproblems. This is in contrast to the static problem of the previous section where the slow and fast subproblems were independent. This interesting feature, which is due to the dynamic nature of the problem, has been noticed elsewhere [34].

The multimodel strategy pair for the dynamic team problem $[\nu_{im}(t,\alpha_i);\ i=1,2]$ is formed by combining the optimal strategies of the slow and fast subproblems

$$\nu_{im}(t,\alpha_i) = \nu^*_{is}(t,\alpha_i) + \tilde{u}^*_{if}(t);\ i=1,2 \quad . \tag{4.52}$$

The following proposition now establishes the well-posedness of this multimodel solution.

Proposition 4.2

a) $\nu^*_i\ (t,\alpha_i) = \nu_{im}\ (t,\alpha_i) + 0(\|\varepsilon\|);\ i=1,2\ ;$

$$t \in [\bar{t}_j, t_{j+1}) \subset [t_j, t_{j+1});\ j \in \theta.$$

b) $J(\nu^*_1,\nu^*_2) = J_s(\nu^*_{1s},\nu^*_{2s}) + \sum_{i=1}^{2} J_{if}\ (\tilde{u}^*_i) + 0(\|\varepsilon\|).$

Proof: If we let

$$\hat{x} = \{\hat{x}'_0\ \hat{x}'_1\ \hat{x}'_2\}' \quad ,$$

where

$$\dot{\hat{x}} = A\hat{x} + B_1\nu_1^*(t,\alpha_1) + B_2\nu_2^*(t,\alpha_2); \quad t \in [t_{j-1}, t_j) ,$$

$$\hat{x}(t_0) = \bar{x}_0; \; j=1, \ldots, N$$

$$\hat{x}(t_j) = \hat{x}(t_j^-) + \hat{K}(j) \, [y(j) - C\hat{x}(t_j^-)] .$$

$$\dot{\Sigma}(t) = A\Sigma + \Sigma A' + FF'; \; t \in [t_{j-1}, t_j) ,$$

$$\Sigma(t_0) = \Sigma_0; \; j-1, \ldots, N$$

$$\Sigma(t_j) = \Sigma(t_j^-) - \hat{K}(j) \, C\Sigma(t_j^-) .$$

$$\hat{K}(j) = \Sigma(t_j^-) \, C'[C\Sigma(t_j^-) \, C' + R_j]^{-1}$$

$$R_j = \mathrm{diag} \, (R_{1j}, R_{2j})$$

$$y(j) = [y_1'(j), \, y_2'(j)]'; \; j=1,2, \ldots, N$$

$$C = [C_1', C_2']'$$

then it is straightforward to notice that

$$\hat{x}_0^*(t) = \hat{\eta}_{0s}^*(t) + 0(\|\epsilon\|)$$

$$\hat{x}_i^*(t) = -A_{ii}^{-1} \, B_{ii} \, \nu_{is}^* \, (t,\alpha_i) + \hat{\eta}_{if}^*(t) + 0(\|\epsilon\|); \; i=1,2 ;$$

$$t \in [\bar{t}_j, t_{j+1}) \subset [t_j, t_{j+1}); \; j \in \theta \quad .$$

The rest of the proof is analogous to the proof of Proposition 4.1 and is therefore omitted here.

The approximation in (a) is valid only on subintervals of the sampling interval due to the fact that we have neglected the boundary-layer terms.

4.5. Conclusions

We have obtained multimodel solutions to LQG team problems under the static information structure and also dynamic information structure with one-step-delay observation-sharing pattern. In both cases the multimodel solution is shown to provide an arbitrarily close approximation to the optimal solution.

The advantages of using the multimodel solution are apparent. Instead of solving one large-dimensioned team problem which is numerically ill-conditioned, the DMs need only solve one low-order team problem, which does not depend on the small uncertain parameters, and two low-order control problems. These control problems can be solved independently by each DM. Hence, each DM need not know the parameters associated with the low-order control problem of the other DM. This implies that the multimodel solution is robust with respect to modeling errors on the part of each DM; a very desirable feature in large scale system design.

The results of this chapter again demonstrate the richness in the modeling structure with multiparameter singular perturbations in the context of multimodeling problems. The limit of seemingly complex integro-differential equations associated with the optimal solution has a nice

appealing structure when rearranged and interpreted as a multimodel solution.

Here we have assumed that the sampling period is fixed and predetermined. If we make the sampling period T a function of the small parameters, such that $T(\varepsilon) \to 0$ as $\|\varepsilon\| \to 0$, then we would not have been able to preserve the one-step-delay observation-sharing pattern in the limit, because the observations become continuous. One way to get around this difficulty would be to make separate observations of the slow and fast variables and let the sampling period of the fast observations be a function of ε. Apparently, this should cause no problems in the asymptotic analysis because the fast subproblems would become continuous stochastic control problems in the limit as $\|\varepsilon\| \to 0$. But it is not clear whether the slow dynamic team problem would require the sharing of the sampled slow observations alone. Of course, in such a case, one will first have to formulate appropriately and solve the dynamic team problem with multirate sampled observations.

From practical considerations, our approach here should cause no problems because the small uncertain parameters are nonzero. This means that in practice the fast variables are not infinitely fast but have a finite bandwidth, and one can always choose an appropriate sampling period from physical considerations.

CHAPTER 5

MULTIMODELING IN MARKOVIAN DECISION PROBLEMS

5.1. Introduction

In the previous chapters we have examined multimodel solutions for two-time-scale systems modeled using multiparameter singular perturbations. In [43,44], the theory of time-scale decomposition has been extended to probabilistic Markov chain models where 'slow' and 'fast' eigenmodes correspond to 'weak' and 'strong' transition probabilities. This chapter focuses on obtaining near-optimal policies for controlled Markov models consisting of N weakly-coupled groups of strongly-interacting states. A hierarchical algorithm, which allows for multimodeling on the part of the decision makers, is proposed for computing these near-optimal policies. The existing results on Markov games [65] do not provide us with a sufficient background to address the multimodeling problem directly. For this reason, we begin by formulating the general N-person average-cost-per-stage problem with state information in Section 5.2. In Section 5.3, the optimality conditions for stationary Feedback Nash and Stackelberg policies are derived. The computational difficulties associated with the feedback policies are discussed in Section 5.4. In Section 5.5 we consider Stackelberg problems when the leader, in addition to the current state of the process, also has access to the followers' controls at every stage [48-50]. An efficient computational algorithm is proposed for computing incentive policies which help the leader to achieve his global optimum.

In Section 5.6, we propose a hierarchical algorithm for computing near-optimal incentive policies for weakly-coupled Markov chains, which allow the 'local' decision makers to use multiple simplified models. Section 5.7 illustrates the well-posedness of the multimodel solution through a numerical example. Finally, the chapter concludes with Section 5.8.

5.2. The N-person Markov Decision Problem with State Information

Consider a finite state Markov chain x_t, $t=0,1,2,\ldots$ with state space $\{1,2,..,n\}$, and controlled by N decision makers with decision variables $\{u_\ell^t,\ \ell=1,2,..,N\}$. The transition probability of the Markov chain at time t depends upon the decisions $\{u_\ell^t;\ \ell=1,2,..,N\}$ chosen at t. Thus, prob $(x_{t+1}|x_t)$ = prob $(x_{t+1}|x_t,u_1^t,..,u_N^t)$. x_t is observed at each t and $\{u_\ell^t;\ \ell=1,2,.,N\}$ may depend on it. Hence, we are concerned with feedback strategies $\{\nu_\ell^t(x(t));\ell=1,2,.,N\}$. If $x_t=i$, then any decision $\{u_\ell^t \in U_\ell(i) \subset \mathbb{R}^{m_\ell};\ \ell=1,2,.,N\}$ may be used. A stationary strategy is any element $\nu\in\Gamma$; $\nu = \{\nu_\ell=(u_\ell(1),u_\ell(2),\ldots,u_\ell(n))\in\Gamma_\ell = U_\ell(1) \times U_\ell(2) \times..\times U_\ell(n),$ $\ell=1,2,.,N\}$, $\Gamma = \{\Gamma_\ell;\ \ell=1,2,.,N\}$. If $x_t=i$, and $\{u_\ell(i);\ \ell=1,2,.,N\}$ is used then

$$P_{ij}(u_1(i),..,u_N(i)) = \text{prob } \{x_{t+1}=j|x_t=i\}$$

where $P_{ij}(u_1(i),\ldots,u_N(i))$: $U_1(i) \times U_2(i)\times..\times U_N(i) \to \mathbb{R}$ are such that

$$P_{ij} \geq 0,\ \sum_j P_{ij} = 1 \quad .$$

For $\{\nu_\ell\in\Gamma_\ell;\ \ell=1,2,.,N\}$, $P(\nu)$ denotes the nxn transition probability matrix

$\{P_{ij}(u_1(i),..,u_N(i)\}$. Note that the i-th row of $P(\nu)$ depends only on $u(i) = \{u_1(i),..,u_N(i)\}$.

Over the long run, each decision maker incurs an expected cost per unit time given by

$$J_\ell(\nu) = \lim_{T\to\infty} \frac{1}{T+1} E \sum_0^T f_\ell\ (x_t,u_1(x_t),..,u_N(x_t))\,;\ \ell=1,2,.,N \quad . \tag{5.1}$$

The following assumption will be in force for the rest of the chapter.

Assumption A:

1) The admissible decision spaces $U_\ell(i)$ are compact.

2) The $P_{ij}(\cdot)$ are continuous functions.

3) For each i, $\{f_\ell(i,\cdot): U_1(i)x..xU_N(i) \to \mathbb{R}\ ;\ \ell=1,2,.,N\}$ are continuous functions.

4) For each $\nu\in\Gamma$, the Markov chain x_t is strongly ergodic.

Assumptions A2 and A4 imply that for each $\nu\in\Gamma$, there is a unique probability row vector $\pi(\nu) = (\pi_1(\nu),.,\pi_n(\nu))$ such that

$$\pi(\nu) = \pi(\nu)\ P(\nu);\ \pi(\nu) > 0\ ; \tag{5.2}$$

furthermore, $\pi(\nu)$ is continuous.

It can be shown [39] that $J_\ell(\nu)$ does not depend on the initial state and is given more simply as,

$$J_\ell(\nu) = \pi(\nu)F_\ell(\nu);\ \ell=1,2,.,N \tag{5.3}$$

where

$$F_\ell(\nu) = [f_\ell(1,u(1)),\ f_\ell(2,u(2)),..,f_\ell(n,u(n))]'$$

Futhermore, $J_\ell(\nu)$ is continuous by Assumption A. It is convenient to introduce the generator

$$Q(\nu) = P(\nu) - I \quad .$$

Then, $Q(\nu) 1_n = 0$; $1_n = (\underbrace{1,1,.,1}_{n})'$ and $\pi(\nu)$ is the unique solution of

$$\pi(\nu)\; Q(\nu) = 0, \; \pi(\nu)\; 1_n = 1 \tag{5.4}$$

The following result is well-known [39].

<u>Lemma 5.1:</u>

Let Assumption A hold. For $\nu \in \Gamma$ consider the linear equations

$$\alpha_\ell 1_n = Q(\nu) C_\ell + F_\ell(\nu); \; \alpha_\ell \in R, \; C_\ell \in R^n; \; \ell = 1,2,.,N \tag{5.5}$$

i) If $\{\alpha_\ell, C_\ell;\ \ell=1,2,.,N\}$ is a solution, then $\alpha_\ell = J_\ell(\nu)$.

ii) If $\{\alpha_\ell, C_\ell;\ \ell=1,2,.,N\}$ is a solution, then so is $\{\alpha_\ell, C_\ell + \delta 1_n;$ $\ell=1,2,.,N\}$ for every δ.

iii) A solution always exists.

Let $Q_i(\nu)$ be the i-th row of $Q(\nu)$. It depends only on $u(i)$. For any C_ℓ let

$$H_\ell(C_\ell, \nu) = Q(\nu) C_\ell + F_\ell(\nu); \; \ell = 1,2,.,N \tag{5.6a}$$

$$H_\ell^i(C_\ell, u(i)) = Q_i(u(i)) C_\ell + f_\ell(i, u(i)); \; \ell = 1,2,.,N \tag{5.6b}$$

5.3. Optimality Conditions for Feedback Nash and Stackelberg Strategies

We shall obtain the optimality conditions for the case N=2 only. The optimality conditions for $N > 2$ can then be obtained in a straightforward manner.

Assumption N:

i) $U_\ell(i)$ is convex; $i=1,2,.,n$; $\ell=1,2$.

ii) For any C_ℓ, $H^i_\ell(C_\ell,u(i))$ is strictly convex in $u_\ell(i)$; $i=1,2,.,n$; $\ell=1,2$.

Assumptions A and N guarantee the existence of the Nash solution.

For the Stackelberg problem we shall assume that DM-1 is the leader and DM-2 is the follower. The following assumption together with Assumption A guarantee the existence of the Stackelberg solution.

Assumption S:

i) $U_2(i)$ is convex; $i=1,2,.,n$.

ii) For any C_2, $H^i_2(C_2,u(i))$ is strictly convex in $u_2(i)$; $i=1,2,.,n$.

Let us consider the Nash solution first. For any (C_1,C_2) define

$$\bar{R}^i_k(u_\ell(i),C_k) = \{\tilde{u}_k(i)\in U_k(i): H^i_k(C_k,u_\ell(i),\tilde{u}_k(i)) = \min_{u_k(i)\in U_k(i)} H^i_k(C_k,u_\ell(i),u_k(i))\}$$

$$u^o_k(i) = \bar{R}^i_k(u_\ell(i),C_k);\ \ell,k=1,2;\ \ell\neq k;\ i=1,2,.,n. \tag{5.7a}$$

$$N^i_\ell(C_1,C_2) = H^i_\ell(C_\ell,u^o_1(i),u^o_2(i));\ i=1,2.,n;\ \ell=1,2 \tag{5.7b}$$

The following theorem gives necessary and sufficient conditions satisfied by the Nash pair.

Theorem 5.1:

Under Assumptions A and N, (ν_1^*, ν_2^*) is Feedback Nash iff there exist $\{(\alpha_\ell^*, C_\ell^*);\ \ell=1,2\}$ such that

$$\alpha_\ell^* 1_n = N_\ell(C_1^*, C_2^*) = H_\ell(C_\ell^*, \nu_1^*, \nu_2^*) \quad .$$

Moreover, $\alpha_\ell^* = J_\ell(\nu_1^*, \nu_2^*) \quad ; \quad \ell=1,2.$

Proof: Follows from Theorem 3.4 of [39] by holding ν_1 fixed in DM-2's optimization problem and vice versa.

We now consider the Stackelberg problem. For each announced strategy $\nu_1 \in \Gamma_1$ of the leader, the follower determines his response by minimizing $J_2(\nu_1, \nu_2)$ over Γ_2. The set of all such solutions

$$R(\nu_1) = \{\nu_2^o \in \Gamma_2 : J_2(\nu_1, \nu_2^o) \leq \min_{\nu_2 \in \Gamma_2} J_2(\nu_1, \nu_2)\} \tag{5.8}$$

is known as the Rational Reaction set of the follower. Assumption S guarantees that $R(\nu_1)$ is a Singleton, and therefore, we have the unique mapping $R: \nu_1 \to \nu_2$. A strategy $\nu_1^* \in \Gamma_1$ is a Stackelberg strategy for the leader if

$$J_1(\nu_1^*, R\nu_1^*) \leq J_1(\nu_1, R\nu_1) \quad ; \quad \forall\ \nu_1 \in \Gamma_1 \quad . \tag{5.9}$$

The optimal strategy for the follower is $\nu_2^* \in R\nu_1^*$. For any (C_1, C_2) define

$$\tilde{R}^i(u_1(i),C_2) = \{\tilde{u}_2(i) \in U_2(i): H_2^i(C_2,u_1(i),\tilde{u}_2(i)) = \min_{u_2(i) \in U_2(i)} H_2^i(C_2,u_1(i), u_2(i))\}$$

$$u_1^o(i) = \arg\min_{u_1(i) \in U_1(i)} H_1^i(C_1,u_1(i),\tilde{R}^i(u_1(i),C_2))$$

$$u_2^o(i) = \tilde{R}^i(u_1^o(i),C_2)$$

$$S_\ell^i(C_1,C_2) = H_\ell^i(C_\ell,u_1^o(i),u_2^o(i)) \quad ; \quad i=1,2.,n; \quad \ell=1,2 . \tag{5.10}$$

The following theorem gives necessary and sufficient conditions satisfied by the Stackelberg strategy.

<u>Theorem 5.2</u>:

Under Assumptions A and S, $(\overset{*}{\nu}_1, \overset{*}{\nu}_2 = R\overset{*}{\nu}_1)$ is Feedback Stackelberg iff there exist $\{(\overset{*}{\alpha}_\ell,\overset{*}{C}_\ell) \;\; ; \;\; \ell=1,2\}$ such that

$$\overset{*}{\alpha}_\ell 1_n = S_\ell(\overset{*}{C}_1,\overset{*}{C}_2) = H_\ell(\overset{*}{C}_\ell,\overset{*}{\nu}_1,\overset{*}{\nu}_2)$$

Moreover, $\overset{*}{\alpha}_\ell = J_\ell(\overset{*}{\nu}_1,\overset{*}{\nu}_2) \quad ; \quad \ell=1,2.$

<u>Proof</u>: i) Sufficiency:

Let there exist $\{(\overset{*}{\alpha}_\ell,\overset{*}{C}_\ell) \;\; ; \;\; \ell=1,2\}$ such that

$$\overset{*}{\alpha}_\ell 1_n = S_\ell(\overset{*}{C}_1,\overset{*}{C}_2) = H_\ell(\overset{*}{C}_\ell,\overset{*}{\nu}_1,\overset{*}{\nu}_2)$$

then,

$$\pi(\overset{*}{\nu}_1,\overset{*}{\nu}_2)\overset{*}{\alpha}_\ell 1_n = \overset{*}{\alpha}_\ell = \pi(\overset{*}{\nu}_1,\overset{*}{\nu}_2)\, H_\ell(\overset{*}{C}_\ell,\overset{*}{\nu}_1,\overset{*}{\nu}_2)$$

$$= \pi(\overset{*}{\nu}_1,\overset{*}{\nu}_2)\, F_\ell(\overset{*}{\nu}_1,\overset{*}{\nu}_2)$$

$$= J_\ell(\overset{*}{\nu}_1,\overset{*}{\nu}_2) \quad .$$

For any $\nu_2 \in \Gamma_2$,

$$\pi(\nu_1^*,\nu_2)\alpha_2^* 1_n \leq \pi(\nu_1^*,\nu_2)\ H_2(C_2^*,\nu_1^*,\nu_2) = J_2(\nu_1^*,\nu_2) \quad .$$

Hence, $\alpha_2^* = J_2(\nu_1^*,\nu_2^*) \leq J_2(\nu_1^*,\nu_2)$.

For any $\nu_1 \in \Gamma_1$, let

$$\tilde{R}(\nu_1,C_2^*) = [\tilde{R}^1(u_1(1),C_2^*),\ \tilde{R}^2(u_1(2),C_2^*),\ldots\tilde{R}^n(u_1(n),C_2^*)]' = R\nu_1$$

$$\pi(\nu_1,R\nu_1)\alpha_1^* 1_n \leq \pi(\nu_1,R\nu_1)\ H_1(C_1^*,\nu_1,R\nu_1) = J_1(\nu_1,R\nu_1).$$

Hence,

$$\alpha_1^* = J_1(\nu_1^*,\nu_2^*) = J_1(\nu_1^*,R\nu_1^*) \leq J_1(\nu_1,R\nu_1)$$

$\Rightarrow$ (ν_1^*,ν_2^*) is a Stackelberg pair.

ii) Necessity:

Suppose $(\nu_1^*,\nu_2^* = R\nu_1^*)$ is Stackelberg.

Let $\{(\alpha_\ell^*,C_\ell^*) \quad ; \quad \ell=1,2\}$ solve

$$\alpha_\ell^* 1_n = Q(\nu_1^*,\nu_2^*)C_\ell^* + F_\ell(\nu_1^*,\nu_2^*) = H_\ell(C_\ell^*,\nu_1^*,\nu_2^*) \quad .$$

Therefore,

$$\alpha_\ell^* = J_\ell(\nu_1^*,\nu_2^*) \quad .$$

Let $\nu_1 \in \Gamma_1$, $\nu_2 = \tilde{R}(\nu_1,C_2^*) = R\nu_1$ be such that

$$H_1(C_1^*,\nu_1,\nu_2) = S_1(C_1^*,C_2^*) \leq \alpha_1^* 1_n \quad .$$

Therefore,

$$J_1(\nu_1,\nu_2) = \pi(\nu_1,\nu_2)\ H_1(C_1^*,\nu_1,\nu_2) \leq \alpha_1^* \quad .$$

Since $\alpha_1^* = J_1(\nu_1^*, \nu_2^*)$, we should have equality above. Hence,

$$\pi(\nu_1, \nu_2)\ [S_1(C_1^*, C_2^*) - \alpha_1^* 1_n] = 0\ .$$

Since $\pi(\nu_1, \nu_2) > 0$ by Assumption A4, we have

$$S_1(C_1^*, C_2^*) = \alpha_1^* 1_n\ .$$

Now, for the follower, let $\nu_2 \in \Gamma_2$ be such that

$$H_2(C_2^*, \nu_1^*, \nu_2) = S_2(C_1^*, C_2^*) \leq \alpha_2^* 1_n\ .$$

Therefore,

$$J_2(\nu_1^*, \nu_2) = \pi(\nu_1^*, \nu_2) H_2(C_2^*, \nu_1^*, \nu_2) \leq \alpha_2^*\ .$$

Since $\alpha_2^* = J_2(\nu_1^*, \nu_2^*)$, we should have equality above.

Hence,

$$\pi(\nu_1^*, \nu_2)[S_2(C_1^*, C_2^*) - \alpha_2^* 1_n] = 0\ .$$

Since $\pi(\nu_1^*, \nu_2) > 0$ by Assumption A4, we have

$$S_2(C_1^*, C_2^*) = \alpha_2^* 1_n\ .$$

Although the above theorems have been proved under the strong ergodicity assumption, it is believed that they hold even when the Markov chain is simply ergodic. The proof for the necessity part without the strong ergodicity assumption will be more involved.

Let us define,

$$\underline{S}_\ell(C_1,C_2) = \min_i S^i_\ell(C_1,C_2)$$

$$\overline{S}_\ell(C_1,C_2) = \max_i S^i_\ell(C_1,C_2)$$

$$\underline{N}_\ell(C_1,C_2) = \min_i N^i_\ell(C_1,C_2)$$

$$\overline{N}_\ell(C_1,C_2) = \max_i N^i_\ell(C_1,C_2) \quad ; \quad \ell=1,2; \tag{5.11}$$

then the following hold:

<u>Lemma 5.2</u>:

For any (C_1,C_2) let (ν_1^o,ν_2^o) be such that

$$S_\ell(C_1,C_2) = H_\ell(C_\ell,\nu_1^o,\nu_2^o) \quad ; \quad \ell=1,2 \ .$$

Then,

$$\underline{S}_\ell(C_1,C_2) \leq J_\ell(\nu_1^o,\nu_2^o) \leq \overline{S}_\ell(C_1,C_2) \quad ; \quad \ell=1,2 \ .$$

(ν_1^o,ν_2^o) is Stackelberg if $\underline{S}_\ell(C_1,C_2) = \overline{S}_\ell(C_1,C_2) \quad ; \quad \ell=1,2.$

<u>Proof</u>: $\underline{S}_\ell(C_1,C_2) \leq \pi(\nu_1^o,\nu_2^o)S_\ell(C_1,C_2) = J_\ell(\nu_1^o,\nu_2^o) \leq \overline{S}_\ell(C_1,C_2)$.

If $\underline{S}_\ell(C_1,C_2) = \overline{S}_\ell(C_1,C_2) = S_\ell(C_1,C_2) = \alpha_\ell 1_n = H_\ell(C_\ell,\nu_1^o,\nu_2^o)$

then (ν_1^o,ν_2^o) is Stackelberg by Theorem 5.2.

<u>Lemma 5.3</u>:

For any (C_1,C_2) let (ν_1^o,ν_2^o) be such that

$$N_\ell(C_1,C_2) = H_\ell(C_\ell,\nu_1^o,\nu_2^o) \quad ; \quad \ell=1,2 .$$

Then,

$$\underline{N}_\ell(C_1,C_2) \le J_\ell(\nu_1^o,\nu_2^o) \le \overline{N}_\ell(C_1,C_2) \quad ; \quad \ell=1,2.$$

(ν_1^o,ν_2^o) is Nash if $\underline{N}_\ell(C_1,C_2) = \overline{N}_\ell(C_1,C_2)$.

Proof: Similar to that of Lemma 5.2.

Notice that unlike the control problem [39], we cannot bound the optimal costs (J_ℓ^* ; $\ell=1,2$) in the Nash and Stackelberg problems by the quantities defined in (5.11). This fact makes it difficult to obtain computational algorithms for the multiple decision maker problems along the lines of control problem [39,40], as we shall see next.

5.4. Computational Aspects

One way to compute the Feedback Nash and Stackelberg policies is to deal directly with equation (5.3) of the cost function. The Nash solution can be computed by obtaining the point of intersection of the reaction curves. The Stackelberg solution can be obtained by applying the algorithm of [51] for static problems. A serious drawback of this direct approach, which makes it computationally infeasible, is that we first need to obtain the steady state distribution $\pi(\nu)$ as a function of $\nu \in \Gamma$. This is very difficult in practice when the Markov chain is of very high dimension and the admissible control sets $U_\ell(i)$ are uncountable.

An alternative approach which does not involve computing $\pi(\nu)$ is to work with dual variables C_ℓ and make use of the results of

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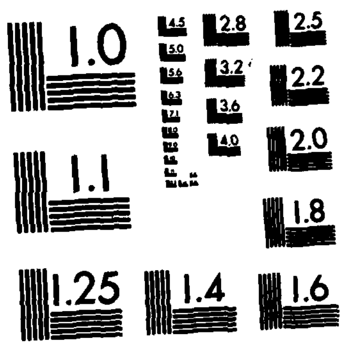

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Lemma 5.1, 5.2, 5.3 and Theorems 5.1, 5.2. The policy iteration [52] and dual variable iteration [39,40] algorithms for the control problem also involve working with linear equations of the type (5.5) rather than computing $\pi(\nu)$ and dealing directly with (5.3).

Let us consider the Nash problem first. For any (C_1^k, C_2^k) we find ν_1^k and ν_2^k from (5.7a) and $N_\ell(C_1^k, C_2^k)$ from (5.7b). We need to update C_ℓ^k such that

$$\lim_{k\to\infty} [\bar{N}_\ell(C_1^k, C_2^k) - \underline{N}_\ell(C_1^k, C_2^k)] = 0 \quad ; \quad \ell=1,2.$$

Then, in the limit, we obtain the Nash solution by Theorem 5.1 and Lemma 5.3. Since we cannot bound J_ℓ^* by $\bar{N}_\ell$ and $\underline{N}_\ell$ at every iteration, the algorithm of [39] cannot be used to update C_1^k and C_2^k. If we do use the algorithm of [39] to update the dual variables, then convergence cannot be guaranteed. But, if the algorithm does converge, then the convergent point is the Nash equilibrium.

If it is known a-priori that the Nash equilibrium is stable [53], then we can use the following policy iteration algorithm which converges to the Nash solution.

<u>Step 1</u>: Choose $\{\nu_\ell^k \in \Gamma_\ell \quad ; \quad \ell=1,2\}$.

<u>Step 2</u>: Obtain ν_ℓ^{k+1} by applying the algorithm of [39] to the following optimization problem.

$$\alpha_\ell^{k+1} 1_n = \min_{\nu_\ell^{k+1} \in \Gamma_\ell} \{Q(\nu_j^k, \nu_\ell^{k+1}) C_\ell^{k+1} + F_\ell(\nu_j^k, \nu_\ell^{k+1})\} \quad ; \quad j,\ell=1,2 \quad ; \quad j\neq\ell \ .$$

<u>Step 3</u>: If $\nu_\ell^{k+1} \approx \nu_\ell^k$ then stop; otherwise set $k \leftarrow k+1$ and go back to Step 2.

We now consider the Stackelberg problem. By the very nature of the problem we cannot have any algorithm based on policy iterations. So any iterative algorithm has to iterate on either one or both dual variables. Consider the following algorithm which involves iterating on both variables. For any (C_1^k, C_2^k) we find (ν_1^k, ν_2^k) and $S_\ell(C_1^k, C_1^k)$ from (5.10). If we can update (C_1^k, C_2^k) such that

$$\lim_{k\to\infty} [\bar{S}_\ell(C_1^k, C_2^k) - \underline{S}_\ell(C_1^k, C_2^k)] = 0 \quad ; \quad \ell=1,2 ;$$

then in the limit we obtain the Stackelberg solution by Theorem 5.2 and Lemma 5.2. Due to the same reason as in the Nash case, we cannot use the algorithm of [39] to update (C_1^k, C_2^k) and guarantee convergence.

It is not possible to develop an algorithm based on updating the leader's dual variable alone. But consider the following algorithm which involves iterating on the follower's dual variable.

<u>Step 1</u>: Choose C_2^k.

<u>Step 2</u>: Find

$$f^k(\nu_1) = \arg\min_{\nu_2 \in \Gamma_2} \{Q(\nu_1, \nu_2)C_2^k + F_2(\nu_1, \nu_2)\}.$$

<u>Step 3</u>: Obtain ν_1^k by applying the algorithm of [39] to the following optimization problem.

$$\alpha_1^k 1_n = \min_{\nu_1^k \in \Gamma_1} \{Q(\nu_1^k, f^k(\nu_1^k))C_1^k + F_1(\nu_1^k, f^k(\nu_1^k))\} .$$

Step 4: Find $\nu_2^k = f^k(\nu_1^k)$, and

$$h(C_2^k) = Q(\nu_1^k,\nu_2^k)C_2^k + F_2(\nu_1^k,\nu_2^k).$$

Step 5: Let $\overline{h}(C_2^k) = \max_i h_i(C_2^k)$

$$\underline{h}(C_2^k) = \min_i h_i(C_2^k) .$$

Update $C_2^k \rightarrow C_2^{k+1}$ such that

$$\overline{h}(C_2^{k+1}) - \underline{h}(C_2^{k+1}) < \overline{h}(C_2^k) - \underline{h}(C_2^k).$$

If $\overline{h}(C_2^k) - \underline{h}(C_2^k) < \delta$, where δ is sufficiently small positive real number, then stop, otherwise

let $k \leftarrow k + 1$ and go back to step 2.

It is very difficult in general to update C_2^k in the desired way because of the implicit dependence of $h(C_2^k)$ on ν_1^k via steps 2 and 3. Due to this dependence, the algorithm of [39] cannot be used to update C_2^k and guarantee convergence. But if we do have convergence, then the limiting solution is Stackelberg by construction.

5.5. Incentive Policies in Stackelberg Problems

We shall now obtain stationary Stackelberg strategies when the leader, in addition to knowing the current state of the process, also has access to the follower's decision variables. Under such an information pattern, the leader has a potential to force the follower to cooperate in achieving his global optimum. Due to the nature of the information pattern,

although the leader declares his strategy first, he actually acts after the follower has made his move at every stage of the game [48]. We shall consider such Stackelberg problems with one leader and N-followers playing Nash, and give an algorithm for computing affine incentive strategy for the leader which helps him achieve his global optimum. Player-0 is assumed to be the leader and players 1, 2,..,N are assumed to be the followers.

The following assumption is made to guarantee a solution to the new Stackelberg problem.

Assumption RS:

i) $U_\ell(i)$ is convex; $i=1,2,.,n$; $\ell=1,2,..,N$

ii) For any C_ℓ, $H^i_\ell(C_\ell, u_o(i), u_1(i),..,u_N(i))$ is strictly convex in $u_o(i)$ and $u_\ell(i)$; $i=1,2,..,n$; $\ell=1,2,..,N$.

The leader's problem is solved in the following steps.

Step 1: Obtain the global optimum of J_o by solving

$$\min_{\nu_o\in\Gamma_o} \min_{\nu_1\in\Gamma_1} \cdots \min_{\nu_N\in\Gamma_N} J_o(\nu_o,\nu_1,.,\nu_N) \tag{5.12}$$

This can be done by applying the algorithm of [39,40]. Denote the minimizing solution by

$$\hat{\nu}_\ell = [\hat{u}_\ell(1),\hat{u}_\ell(2),\ldots,\hat{u}_\ell(n)] \quad ; \quad \ell=0,1,.,N.$$

Step 2: Choose the leader's strategy as

$$\nu_o^* = \hat{\nu}_o - \sum_{\ell=1}^{N} P_\ell[\nu_\ell-\hat{\nu}_\ell] \tag{5.13}$$

where

$$P_{\ell} = \text{diag}\,[P_{\ell 1},\ P_{\ell 2},..,P_{\ell n}] \quad ; \quad \ell=1,2,..,N.$$

This strategy has the open-loop value $\nu_o^* = \hat{\nu}_o$ whenever the followers are forced to play $\{\nu_{\ell}^* = \hat{\nu}_{\ell} \;\; ; \;\; \ell=1,2,..,N\}$. Since $(\hat{\nu}_o,\ \hat{\nu}_1,..,\hat{\nu}_N)$ is the desired open-loop solution for the leader, the P_{ℓ} are chosen such that the followers' optimal reaction is $\{\nu_{\ell}^* = \hat{\nu}_{\ell};\quad \ell=1,2,..,N\}$.

Step 3: Solve the linear equations for $\{(\alpha_{\ell}^*, C_{\ell}^*) \quad ; \quad \ell=1,2,..,N\}$

$$\alpha_{\ell}^* 1_n = Q(\hat{\nu}_o,\hat{\nu}_1,..,\hat{\nu}_N)C_{\ell}^* + F_{\ell}(\hat{\nu}_o,\hat{\nu}_1,..,\hat{\nu}_N) \quad ; \quad \ell=1,2,..,N \quad .$$

Step 4: Obtain $\{P_{\ell};\ \ell=1,2,..,N\}$ from the gradient equations of [50], which in our case can be written as,

$$P_{\ell i}\cdot\nabla_{u_o(i)}H_{\ell}^i(C_{\ell}^*,\hat{u}_o(i),\hat{u}_1(i),...,\hat{u}_N(i)) = \nabla_{u_{\ell}(i)}H_{\ell}^i(C_{\ell}^*,\hat{u}_o(i),\hat{u}_1(i),...,\hat{u}_N(i))$$

$$i=1,2,..,n; \quad \ell=1,2,..,N.$$

The leader declares to follower-ℓ, ν_o^* and $\{\hat{\nu}_j;\quad j=1,..,N;\ \ j \neq \ell\}$. Then, follower-$\ell$ solves the optimization problem

$$\alpha_{\ell} 1_n = \min_{\nu_{\ell}\in\Gamma_{\ell}} \{Q(\nu_o^*,\hat{\nu}_1,..,\hat{\nu}_{\ell-1},\nu_{\ell},\hat{\nu}_{\ell+1},..,\hat{\nu}_N)C_{\ell} + F_{\ell}(\nu_o^*,\hat{\nu}_1,...,\hat{\nu}_{\ell-1},\nu_{\ell},\hat{\nu}_{\ell+1},...,\hat{\nu}_N)\}$$

and obtains his optiaml strategy $\nu_{\ell}^* = \hat{\nu}_{\ell}$ by applying the algorithm of [39,40].

Notice that the Stackelberg solution of this section is computationally easier to obtain than the Stackelberg solution of the previous section.

5.6. Incentive Policies for Weakly-Coupled Markov Chains

Now we shall consider the incentive design problem in the context of large Markov chains consisting of N weakly-coupled groups of strongly-interacting states. Such models arise naturally in the modeling of reservoir dynamics in hydro-scheduling problems [41,42] and queueing network models of computer systems [46,47]. We shall assume that transitions from each group are controlled by a single decision maker having his own performance objective and the overall system is coordinated by a leader whose objective is to optimize some global system performance. The computational algorithm for obtaining the near-optimal policies will be shown to exhibit multimodel features, i.e., each lower level decision maker, in order to compute his near-optimal policy, need only know his 'local' dynamics and some 'aggregate' of the rest of the system.

Weakly-coupled Markov chains are described by the generator matrix $A + \varepsilon B$ [43,44], where A and B are both n-dimensional Markov generators having the form

$$A = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_N \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix} \qquad (5.14)$$

with $\{A_j,\ j=1,2,..,N\}$ being n_j-dimensional Markov generators.

Thus the Markov chain consists of N groups of strongly-interacting states. The weak interactions between states in different groups are modeled as multiples of a small positive scalar ε.

For the decision problem to be considered we have

$$A_j = A_j(\nu_o^j,\nu_j) \;,\quad B_j = B_j(\nu_o^j,\nu_j)$$

$$\nu_o^j = [u_o(n_1+..+n_{j-1}+1),\ldots,u_o(n_1+..+n_j)] \quad ; \quad \nu_o = [\nu_o^1\nu_o^2..\nu_o^N]$$

$$\nu_j = [u_j(n_1+..+n_{j-1}+1),..,u_j(n_1+..+n_j)] \quad ; \quad j=1,2,.,N \tag{5.15}$$

where, as before, player 0 is the leader and players 1,2,.,N are the followers playing Nash.

The cost vectors of the decision makers are of the form

$$F_\ell(\nu_o,\nu_1,.,\nu_N) = \begin{bmatrix} f_\ell(1,u_1(1),u_o(1)) \\ \vdots \\ f_\ell(n_1,u_1(n_1),u_o(n_1)) \\ \hline f_\ell(n_1+1,u_2(n_1+1),u_o(n_1+1)) \\ \vdots \\ f_\ell(n_1+n_2,u_2(n_1+n_2),u_o(n_1+n_2)) \\ \hline \vdots \\ \hline f_\ell(n-n_N+1,u_N(n-n_N+1),u_o(n-n_N+1)) \\ \vdots \\ f_\ell(n,u_N(n),u_o(n)) \end{bmatrix} = \begin{bmatrix} F_{\ell 1}(\nu_o^1,\nu_1) \\ \hline F_{\ell 2}(\nu_o^2,\nu_2) \\ \hline \vdots \\ \hline F_{\ell N}(\nu_o^N,\nu_N) \end{bmatrix}$$

$$\ell=0,1,2,.,N. \tag{5.16}$$

The following assumption is made about the process.

<u>Assumption B</u>:

1) For all $\{\nu_\ell \in \Gamma_\ell;\ \ell=0,1,.,N\}$ and $0 < \varepsilon \leq \varepsilon^*$, the Markov process defined by $A + \varepsilon B$ has a single ergodic class.

2) For each $\nu_o \in \Gamma_o$ and $\nu_j \in \Gamma_j$, the Markov process defined by $A_j(\nu_o^j, \nu_j)$ has a single ergodic class.

Assumption B2 implies that each $A_j(\nu_o^j, \nu_j)$ has one zero eigenvalue. The corresponding right eigenvector t_j is the n_j-dimensional column made of ones. The left eigenvector $v_j(\nu_o^j, \nu_j)$ is the n_j-dimensional row of stationary probabilities for the states in the j-th group when $\varepsilon=0$.

Let,

$$T = \begin{bmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_N \end{bmatrix} \quad ; \quad V(\nu_o, \nu_1, .., \nu_N) = \begin{bmatrix} v_1(\nu_o^1, \nu_1) & & & 0 \\ & v_2(\nu_o^2, \nu_2) & & \\ & & \ddots & \\ 0 & & & v_N(\nu_o^N, \nu_N) \end{bmatrix} \tag{5.17}$$

It has been shown in [44] that the n-dimensional probability vector p of the Markov process can be approximated by,

$$p = \pi V + 0(\varepsilon) \tag{5.18}$$

where π is the N-dimensional probability vector of the aggregate Markov process with generator VBT describing the transitions between different groups.

Let $\pi(\nu,\varepsilon)$ and $\bar{\pi}(\nu)$ be the unique solutions of

$$\pi(\nu,\varepsilon)\ [A(\nu) + \varepsilon B(\nu)] = 0 \quad ; \quad \pi(\nu,\varepsilon)1_n = 1 \ , \tag{5.19}$$

$$\bar{\pi}(\nu)\ V(\nu)B(\nu)T = 0 \quad ; \quad \bar{\pi}(\nu)1_N = 1 \ , \tag{5.20}$$

where $\nu = (\nu_o, \nu_1, ., \nu_N)$.

Then we have,

$$\pi(\nu,\varepsilon) = \bar{\pi}(\nu)\ V(\nu) + O(\varepsilon) \quad . \tag{5.21}$$

For any given policy $\nu \in \Gamma$, the average cost per stage can be approximated as,

$$\begin{aligned} J_\ell(\nu,\varepsilon) &= \pi(\nu,\varepsilon)\ F_\ell(\nu) \\ &= \bar{\pi}(\nu)\ V(\nu)\ F_\ell(\nu) + O(\varepsilon) \\ &= \bar{J}_\ell(\nu) + O(\varepsilon) \quad ; \quad \ell=0,1,2,.,N. \end{aligned} \tag{5.22}$$

$\bar{J}_\ell(\nu)$ is the average cost per stage associated with the aggregate chain and

$$\bar{J}_\ell(\nu) = \bar{\pi}(\nu)\ \bar{F}_\ell(\nu) \quad ; \quad \ell=0,1,2,.,N. \tag{5.23}$$

where $\bar{F}_\ell(\nu) = V(\nu)F_\ell(\nu)$ is the N-dimensional instantaneous cost vector associated with the aggregate chain.

We shall now obtain near-optimal policies based on the aggregate costs $\bar{J}_\ell(\nu)$. In terms of the aggregate dual variables $\bar{C}_\ell$ we can write

$$\bar{\alpha}_\ell 1_N = V(\nu)\ B(\nu)T\ \bar{C}_\ell + \bar{F}_\ell(\nu)$$

$$\bar{\alpha}_\ell = \bar{J}_\ell(\nu) \quad ; \quad \bar{C}_\ell \in \mathbb{R}^N \quad ; \quad \ell=0,1,2,..,N \quad . \qquad (5.24)$$

Alternatively,

$$\bar{\alpha}_\ell 1_N = V(\nu)\ [B(\nu)T\bar{C}_\ell + F_\ell(\nu)]$$

$$= V(\nu)\ g_\ell(\nu) \quad ; \quad \ell=0,1,2,..,N; \qquad (5.25)$$

where

$$g_\ell(\nu) = \begin{bmatrix} g_{\ell 1}(\nu_o^1,\nu_1) \\ g_{\ell 2}(\nu_o^2,\nu_2) \\ \vdots \\ g_{\ell N}(\nu_o^N,\nu_N) \end{bmatrix} = \begin{bmatrix} B_1(\nu_o^1,\nu_1)T\bar{C}_\ell + F_{\ell 1}(\nu_o^1,\nu_1) \\ B_2(\nu_o^2,\nu_2)T\bar{C}_\ell + F_{\ell 2}(\nu_o^2,\nu_2) \\ \vdots \\ B_N(\nu_o^N,\nu_2 T\bar{C}_\ell + F_{\ell N}(\nu_o^N,\nu_N) \end{bmatrix} . \qquad (5.26)$$

Therefore, in component form (5.25) becomes

$$\bar{\alpha}_\ell = v_j(\nu_o^j,\nu_j)\ g_{\ell j}(\nu_o^j,\nu_j) \quad ; \quad j=1,2,..,N \quad ; \quad \ell=0,1,2,..,N \quad . \qquad (5.27)$$

Each component in (5.27) can be interpreted as average cost per stage associated with the n_j-dimensional local chains with generators $A_j(\nu_o^j,\nu_j)$.

Hence, we can write

$$\bar{\alpha}_{\ell} 1_{n_j} = A_j(\nu_o^j,\nu_j)C_{\ell j} + g_{\ell j}(\nu_o^j,\nu_j) \quad ; \quad j=1,2,.,N \quad ; \quad \ell=0,1,2,.,N \; ; \tag{5.28}$$

where $C_{\ell j} \in \mathbb{R}^{nj}$ are the dual variables associated with the local chains.

Based on the hierarchical structure given by (5.24)-(5.28) of the aggregate costs, we now formulate an algorithm to solve the leader's problem.

Step 1: Obtain the global optimum of $\bar{J}_o(\nu)$ by the following iterative scheme.

i) Choose $\bar{C}_o^k$

ii) Solve

$$h_{oj}(\bar{C}_o^k)1_{n_j} = \min_{\nu_o^j \in \Gamma_o^j} \quad \min_{\nu_j \in \Gamma_j} \quad [A_j(\nu_o^j,\nu_j)C_{oj}^k + g_{oj}^k(\nu_o^j,\nu_i)] \; ; \; j=1,2,.,N \; ;$$

using the algorithm of [39,40]

iii) Find $\bar{h}_o(\bar{C}_o^k) = \max_j h_{oj}(\bar{C}_o^k)$

$$\underline{h}_o(\bar{C}_o^k) = \min_j h_{oj}(\bar{C}_o^k)$$

If $\bar{h}_o(\bar{C}_o^k) - \underline{h}_o(\bar{C}_o^k) \approx 0$, then stop; otherwise update $\bar{C}_o^k$ by the algorithm of [39]; set $k \leftarrow k+1$ and return to (ii). Denote the optimal solution by $\hat{\nu}_{\ell}$; $\ell=0,1,2,.,N$.

Step 2: Solve the linear equations

$$\bar{\alpha}_{\ell} 1_N = V(\hat{\nu})B(\hat{\nu})T\bar{C}_{\ell} + \bar{F}_{\ell}(\hat{\nu})$$

$$\bar{\alpha}_{\ell} 1_{n_\ell} = A_{\ell}(\hat{\nu}_o^{\ell},\hat{\nu}_{\ell})C_{\ell\ell} + g_{\ell\ell}(\hat{\nu}_o^{\ell},\hat{\nu}_{\ell}) \quad ; \quad \ell=1,2,.,N.$$

Step 3: Choose the leader's strategy as

$$\tilde{\nu}_o^{\ell} = \hat{\nu}_o^{\ell} - P_{\ell}[\nu_{\ell} - \hat{\nu}_{\ell}] \quad ; \quad P_{\ell} = \text{diag}\,[P_{\ell 1}, P_{\ell 2},..,P_{\ell n_\ell}]$$

and solve for P_{ℓ} from the gradient equations

$$P_{\ell}'\nabla_{\nu_o^{\ell}} \bar{H}_{\ell}(C_{\ell\ell},\hat{\nu}_o^{\ell},\hat{\nu}_{\ell}) = \nabla_{\nu_{\ell}} \bar{H}_{\ell}(C_{\ell\ell},\hat{\nu}_o^{\ell},\hat{\nu}_{\ell})$$

$$\bar{H}_{\ell}(C_{\ell\ell},\hat{\nu}_o^{\ell},\hat{\nu}_{\ell}) = A_{\ell}(\hat{\nu}_o^{\ell},\hat{\nu}_{\ell})C_{\ell\ell} + g_{\ell\ell}(\hat{\nu}_o^{\ell},\hat{\nu}_{\ell})$$

$$\ell=1,2,.,N .$$

The leader declares to follower-ℓ, $\tilde{\nu}_o$ and $\{\hat{\nu}_j;\ j=1,.,N;\ j\neq\ell\}$. Follower-$\ell$ then solves his optimization problem by the following iterative scheme.

Step 1: Choose $\bar{C}_{\ell}^k$

Step 2: Compute

$$h_{\ell j}(\bar{C}_{\ell}^k) = v_j(\hat{\nu}_o^j,\hat{\nu}_j)\ [B_j(\hat{\nu}_o^j,\hat{\nu}_j)T\bar{C}_{\ell}^k + F_{\ell j}(\hat{\nu}_o^j,\hat{\nu}_j)]$$

$$= v_j(\hat{\nu}_o^j,\hat{\nu}_j)\ g_{\ell j}^k(\hat{\nu}_o^j,\hat{\nu}_j) \quad ; \quad j=1,2,.,N;\ j\neq\ell.$$

Step 3: Solve the local optimization problem

$$h_{\ell\ell}(\overline{C}^k_\ell)1_{n_\ell} = \min_{\nu_\ell \in \Gamma_\ell} [A_\ell(\tilde{\nu}^\ell_o, \nu_\ell)C^k_{\ell\ell} + g^k_{\ell\ell}(\tilde{\nu}^\ell_o, \nu_\ell)]$$

applying the algorithm of [39].

Step 4: Find

$$\overline{h}_\ell(\overline{C}^k_\ell) = \max_j h_{\ell j}(\overline{C}^k_\ell)$$

$$\underline{h}_\ell(\overline{C}^k_\ell) = \min_j h_{\ell j}(\overline{C}^k_\ell)$$

If $\overline{h}_\ell(\overline{C}^k_\ell) - \underline{h}_\ell(\overline{C}^k_\ell) \approx 0$ then stop; otherwise update $\overline{C}^k_\ell$ by the algorithm of [39]; set $k \leftarrow k+1$ and return to step 2.

When the algorithm converges, the leader's declared strategy ensures that the followers' optimal reaction based on the aggregate costs would be $\{\tilde{\nu}_\ell = \hat{\nu}_\ell \; ; \; \ell=1,2,..,N\}$.

Let us now examine the salient features of the algorithm presented above. Specifically, we would like to see what each decision maker has to know about the system model and the costs in order to compute his strategies. The leader, being the overall coordinator has to know the full A and B matrices and the cost vectors of all the decision makers. Each follower on the other hand, need only know his own local generator matrix A_ℓ, the interconnection matrix B and the steady state distribution of the other local Markov chains along the optimal solution. He need not know the detailed dynamics of the other local Markov chains. This multimodel situation accounts for many practical problems where the 'local'

decision makers do not have an exact knowledge of the 'global' model. Note that none of the decision makers need to know the value of the perturbation parameter ε.

In the sequel we shall give a series of propositions which establish the asymptotic properties of the multimodel solution given above.

Let us denote the optimal Stackelberg solution for the full problem as

$$\nu_o^*(\varepsilon),\ \nu_1^*(\varepsilon),\ .\ .\ ,\ \nu_N^*(\varepsilon) \qquad .$$

The following proposition establishes the 'closeness' of the multimodel solution to the optimal solution.

Proposition 5.1:

If the multimodel solution $\{\hat{\nu}_o,\ \hat{\nu}_1,\ .\ .\ ,\ \hat{\nu}_N\}$ is unique, then

i) $\quad J_o(\hat{\nu}_o,\ \hat{\nu}_1,\ .\ .\ ,\ \hat{\nu}_N) - J_o(\nu_o^*,\ \nu_1^*,\ .\ .\ ,\ \nu_N^*) = O(\varepsilon^2)$.

ii) $\quad \nu_\ell^* = \hat{\nu}_\ell + O(\varepsilon) \quad ; \quad \ell=0,1,2,\ .\ .\ ,\ N$.

iii) $\quad J_\ell(\hat{\nu}_o,\ \hat{\nu}_1,\ .\ .\ ,\ \hat{\nu}_N) - J_\ell(\nu_o^*,\ \nu_1^*,\ .\ .\ ,\ \nu_N^*) = O(\varepsilon) \quad ; \quad \ell=1,2,\ .\ .\ ,\ N$.

Proof: (i) and (ii) follow directly from [44] because the leader's problem is a global minimization problem.

To prove (iii) we let

$$J_\ell(\hat{\nu}) - J_\ell(\nu^*) = J_\ell(\hat{\nu}) - \bar{J}_\ell(\hat{\nu}) - J_\ell(\nu^*) + \bar{J}_\ell(\nu^*) + \bar{J}_\ell(\hat{\nu}) - \bar{J}_\ell(\nu^*)$$

$$\le |J_\ell(\hat{\nu}) - \bar{J}_\ell(\hat{\nu})| + |J_\ell(\nu^*) - \bar{J}_\ell(\nu^*)| + |\bar{J}_\ell(\hat{\nu}) - \bar{J}_\ell(\nu^*)|$$

Due to (5.22) we have

$$|J_\ell(\hat{\nu}) - \bar{J}_\ell(\hat{\nu})| = 0(\varepsilon)$$

$$|J_\ell(\nu^*) - \bar{J}_\ell(\nu^*)| = 0(\varepsilon) \quad .$$

Due to (ii) and the continuity of the aggregate costs in the controls we have $|\bar{J}_\ell(\hat{\nu}) - \bar{J}_\ell(\nu^*)| = 0(\varepsilon)$.

Hence, (iii) follows.

The following two propositions establish the robustness properties of the multimodel solution.

<u>Proposition 5.2</u>:

Let $\bar{\nu}_\ell = \arg\min_{\nu_\ell \in \Gamma_\ell} J_\ell(\tilde{\nu}_o, \hat{\nu}_1, \ldots, \nu_\ell, \ldots, \hat{\nu}_N)$; $\ell=1,2,\ldots,N$;

then ,

i) $J_\ell(\tilde{\nu}_o, \hat{\nu}_1, \ldots, \hat{\nu}_N) - J_\ell(\tilde{\nu}_o, \hat{\nu}_1, \ldots, \bar{\nu}_\ell, \ldots, \hat{\nu}_N) = 0(\varepsilon^2)$; $\ell=1,2,\ldots,N$

i.e.; no follower can benefit significantly by deviating unilaterally from the multimodel solution.

ii) $J_k(\tilde{\nu}_o, \hat{\nu}_1, \ldots, \hat{\nu}_N) - J_k(\tilde{\nu}_o, \hat{\nu}_1, \ldots, \bar{\nu}_\ell, \ldots, \hat{\nu}_N) = 0(\varepsilon)$; $k=0,1,\ldots,N$; $k\neq\ell$

i.e.; by deviating unilaterally from the multimodel solution, no follower can hurt the other decision makers significantly.

<u>Proof</u>:

i) Since $\hat{\nu}_\ell = \arg\min_{\nu_\ell \in \Gamma_\ell} \bar{J}_\ell(\tilde{\nu}_o, \hat{\nu}_1, \ldots, \nu_\ell, \ldots, \hat{\nu}_N)$, it follows directly from [44] that

$$J_\ell(\tilde{\nu}_o, \hat{\nu}_1, \ldots, \hat{\nu}_N) - J_\ell(\tilde{\nu}_o, \hat{\nu}_1, \ldots, \bar{\nu}_\ell, \ldots, \hat{\nu}_N) = 0(\varepsilon^2)$$

Furthermore, $\bar{\nu}_\ell = \hat{\nu}_\ell + 0(\varepsilon)$; $\ell=1,2,\ldots,N$.

ii) $J_k(\tilde{\nu}_o,\hat{\nu}_1, \cdot\,, \hat{\nu}_N) - J_k(\tilde{\nu}_o, \hat{\nu}_1, \cdot\,, \bar{\nu}_\ell, \cdot\,, \hat{\nu}_N) \quad ; \quad k \neq \ell$

$= \bar{J}_k(\tilde{\nu}_o,\hat{\nu}_1, \cdot\,, \hat{\nu}_N) - \bar{J}_k(\tilde{\nu}_o, \hat{\nu}_1, \cdot\,, \bar{\nu}_\ell, \cdot\,, \hat{\nu}_N) + O(\epsilon)$

$= O(\epsilon)$, due to (i) and the continuity of the aggregate costs in the controls.

<u>Proposition 5.3</u>:

Let $(\bar{\nu}_1,\bar{\nu}_2, \cdot\,, \bar{\nu}_N)$ be the optimal Nash reaction of the followers to the declared strategy $\tilde{\nu}_o$ of the leader; then

i) $J_o(\tilde{\nu}_o,\hat{\nu}_1, \cdot\,, \hat{\nu}_N) - J_o(\tilde{\nu}_o,\bar{\nu}_1,\bar{\nu}_2, \cdot\,, \bar{\nu}_N) = O(\epsilon)$

i.e.; the leader does not lose significantly if the followers, instead of playing their multimodel strategies, respond optimally.

ii) $J_o(\tilde{\nu}_o,\bar{\nu}_1,\bar{\nu}_2, \cdot\,, \bar{\nu}_N) - J_o(\nu_o^*,\nu_1^*, \cdot\,, \nu_N^*) = O(\epsilon)$

i.e.; the leader does not lose significantly by declaring his multimodel strategy $\tilde{\nu}_o$ instead of his optimal strategy ν_o^*.

<u>Proof</u>:

i) Define

$$J_\ell^o = J_\ell(\tilde{\nu}_o, \cdot, \ldots, \cdot) \quad ; \quad \ell=0,1,2, \cdot\,, N.$$

By observing that $\{\bar{\nu}_\ell \;;\; \ell=1,2, \cdot, N\}$ is the optimal Nash solution for the followers with respect to the costs $\{J_\ell^o \;;\; \ell=1,2, \cdot\,, N\}$; and $\{\hat{\nu}_\ell \;;\; \ell=1,2, \cdot\,, N\}$ is the optimal Nash solution for the followers with respect to the aggregate costs $\{\bar{J}_\ell^o \;;\; \ell=1,2, \cdot\,, N\}$; we can show by constructing matched asymptotic expansions as in [44]

that

$$J_{\ell}^{o}(\hat{\nu}_1, \ldots, \hat{\nu}_N) = J_{\ell}^{o}(\bar{\nu}_1, \ldots, \bar{\nu}_N) \quad ; \quad \ell=1,2, \ldots, N ;$$

furthermore, $\bar{\nu}_{\ell} = \hat{\nu}_{\ell} + O(\varepsilon) \quad ; \quad \ell=1,2, \ldots, N.$

Hence, (i) follows because of continuity of costs.

ii) follows from (i) above and (i) of Proposition 5.1.

5.7. An Example

We shall now consider a numerical example of a weakly-coupled Markov chain and obtain the near-optimal incentive policies. The example is motivated by the following hydro-scheduling problem for electric power generation.

Consider a hydro-power system consisting of a central reservoir which feeds into three local reservoirs. For simplicity assume that the central reservoir feeds into the local reservoirs one at a time, and switches between reservoirs in a random fashion.

When the central reservoir is feeding into one of the local reservoirs, the other two reservoirs are assumed to be in some 'idle' state. Each local reservoir is assumed to be under the authority of a separate decision maker who controls the rate of water release u_i for electric power generation. The 'state' of each local reservoir is characterized by its water level, assumed to be 1,2,3, when it is active, and 'idle' when it is inactive. The central reservoir is assumed to have an infinite capacity. The local level changes are assumed to be of high probability compared to the switching of the central reservoir between the

different local reservoirs. There is an overall coordinator or leader who controls the rate of switching and the inflows into the local reservoirs. His decision variable is assumed to be u_o. The objective of each local decision maker is to minimize his own local average production cost per unit time, whereas the objective of the leader is to minimize the global average production cost per unit time.

The above system can be modeled by a nine state Markov chain consisting of 3 weakly-coupled groups of strongly-interacting states. The states are as follows:

1 = (inflow into reservoir 1, level 1, reservoirs 2,3 idle)

2 = (inflow into reservoir 1, level 2, reservoirs 2,3 idle)

3 = (inflow into reservoir 1, level 3, reservoirs 2,3 idle)

4 = (inflow into reservoir 2, level 1, reservoirs 1,3 idle)

5 = (inflow into reservoir 2, level 2, reservoirs 1,3 idle)

6 = (inflow into reservoir 2, level 3, reservoirs 1,3 idle)

7 = (inflow into reservoir 3, level 1, reservoirs 1,2 idle)

8 = (inflow into reservoir 3, level 2, reservoirs 1,2 idle)

9 = (inflow into reservoir 3, level 3, reservoirs 1,2 idle).

The system matrices are assumed to be the following:

$$A_i(u_o,u_i) = \begin{bmatrix} -0{\cdot}08-0{\cdot}1u_i+0{\cdot}1u_o & 0{\cdot}08+0{\cdot}1u_i-0{\cdot}1u_o & 0 \\ 0{\cdot}05-0{\cdot}06u_i+0{\cdot}065u_o & -0{\cdot}1-0{\cdot}01u_i-0{\cdot}005u_o & 0{\cdot}05+0{\cdot}07u_i-0{\cdot}06u_o \\ 0 & 0{\cdot}09+0{\cdot}1u_o-0{\cdot}02u_i & -0{\cdot}09-0{\cdot}1u_o+0{\cdot}02u_i \end{bmatrix}$$

$$i=1,2,3$$

$$B_{ij}(u_o) = \begin{bmatrix} 0 & 0 & 0{\cdot}2u_o \\ 0 & 0 & 0{\cdot}1u_o \\ 0 & 0 & 0{\cdot}15u_o \end{bmatrix} \quad , \quad i,j=1,2,3 \quad ; \quad i \neq j$$

$$B_{ii}(u_o) = \begin{bmatrix} -0{\cdot}4u_o & 0 & 0 \\ 0 & -0{\cdot}2u_o & 0 \\ 0 & 0 & -0{\cdot}3u_o \end{bmatrix} \quad , \quad i=1,2,3 \quad .$$

The control sets are

$$U_o = \{u_o : 0{\cdot}01 \le u_o \le 0{\cdot}1\}$$

$$U_\ell = \{u_\ell : 0{\cdot}05 \le u_\ell \le 0{\cdot}2\} \quad ; \quad \ell=1,2,3.$$

The instantaneous costs are ,

$$f_1(n,u_1(n),u_o(n)) = (2-n)^2 + 25(u_1(n))^2 + 10(u_o(n))^2 \quad ; \quad n=1,2,3$$

$$f_2(n,u_o(n),u_2(n)) = (5-n)^2 + 30(u_1(n))^2 + 10(u_o(n))^2 \quad ; \quad n=4,5,6$$

$$f_3(n,u_o(n),u_3(n)) = (8-n)^2 + 20(u_3(n))^2 + 15(u_o(n))^2 \quad ; \quad n=7,8,9$$

$$f_o(n,u_o(n),u_1(n),u_2(n),u_3(n)) = \frac{1}{3}\sum_{i=1}^{3} f_i(n,u_o(n),u_i(n))^2 \quad .$$

Using the algorithm of the previous section, the near-optimal affine incentive policy for the leader is obtained as

$$\tilde{\nu}_o = \begin{bmatrix} \tilde{u}_o(1) \\ \tilde{u}_o(2) \\ \tilde{u}_o(3) \\ \tilde{u}_o(4) \\ \tilde{u}_o(5) \\ \tilde{u}_o(6) \\ \tilde{u}_o(7) \\ \tilde{u}_o(8) \\ \tilde{u}_o(9) \end{bmatrix} = \begin{bmatrix} 0{\cdot}067 - 0{\cdot}5833(u_1(1) - 0{\cdot}146) \\ 0{\cdot}052 - 0{\cdot}4762(u_1(2) - 0{\cdot}098) \\ 0{\cdot}046 - 0{\cdot}6334(u_1(3) - 0{\cdot}051) \\ 0{\cdot}071 - 0{\cdot}5721(u_2(4) - 0{\cdot}131) \\ 0{\cdot}066 - 0{\cdot}4654(u_2(5) - 0{\cdot}081) \\ 0{\cdot}056 - 0{\cdot}6142(u_2(6) - 0{\cdot}051) \\ 0{\cdot}055 - 0{\cdot}6518(u_3(7) - 0{\cdot}164) \\ 0{\cdot}048 - 0{\cdot}5532(u_3(8) - 0{\cdot}112) \\ 0{\cdot}044 - 0{\cdot}7156(u_3(9) - 0{\cdot}06) \end{bmatrix}$$

The optimal strategies for the followers' are given by,

$$\hat{\nu}_1 = \begin{bmatrix} \hat{u}_1(1) \\ \hat{u}_1(2) \\ \hat{u}_1(3) \end{bmatrix} = \begin{bmatrix} 0\cdot146 \\ 0\cdot098 \\ 0\cdot051 \end{bmatrix}, \quad \hat{\nu}_2 = \begin{bmatrix} \hat{u}_2(4) \\ \hat{u}_2(5) \\ \hat{u}_2(6) \end{bmatrix} = \begin{bmatrix} 0\cdot131 \\ 0\cdot081 \\ 0\cdot051 \end{bmatrix},$$

$$\hat{\nu}_3 = \begin{bmatrix} \hat{u}_3(7) \\ \hat{u}_3(8) \\ \hat{u}_3(9) \end{bmatrix} = \begin{bmatrix} 0\cdot164 \\ 0\cdot112 \\ 0\cdot06 \end{bmatrix}.$$

The resulting costs (for $\varepsilon = 0\cdot1$) are given by

$$\hat{J}_1 = 0\cdot76541$$

$$\hat{J}_2 = 0\cdot75332$$

$$\hat{J}_3 = 0\cdot75884$$

$$\hat{J}_o = 0\cdot75917$$

We now compare the near-optimal solution $\tilde{\nu}_o$ to the optimal solution for $\varepsilon = 0.5$, 0.1, 0.01.

$\varepsilon = 0.5$
$0.052 - 0.5133(u_1(1) - 0.132)$
$0.041 - 0.4521(u_1(2) - 0.09)$
$0.041 - 0.5964(u_1(3) - 0.053)$
$0.075 - 0.5548(u_2(4) - 0.12)$
$0.068 - 0.4131(u_2(5) - 0.062)$
$0.052 - 0.5861(u_2(6) - 0.05)$
$0.066 - 0.6142(u_3(7) - 0.161)$
$0.051 - 0.5091(u_3(8) - 0.092)$
$0.048 - 0.6634(u_3(9) - 0.053)$
$J_o^* = 0.73718$
$J_o(\hat{\nu}) = 0.78213$

$\varepsilon = 0.1$
$0.061 - 0.5622(u_1(1) - 0.14)$
$0.048 - 0.4702(u_1(2) - 0.102)$
$0.045 - 0.6208(u_1(3) - 0.051)$
$0.072 - 0.5637(u_2(4) - 0.124)$
$0.066 - 0.4543(u_2(5) - 0.08)$
$0.055 - 0.6004(u_2(6) - 0.051)$
$0.058 - 0.6427(u_3(7) - 0.164)$
$0.048 - 0.5324(u_3(8) - 0.1)$
$0.045 - 0.6988(u_3(9) - 0.055)$
$J_o^* = 0.75012$
$J_o(\hat{\nu}) = 0.75917$

$\varepsilon = 0.01$
$0.066 - 0.5848(u_1(1) - 0.144)$
$0.052 - 0.4814(u_1(2) - 0.1)$
$0.045 - 0.6444(u_1(3) - 0.051)$
$0.07 - 0.5688(u_2(4) - 0.13)$
$0.068 - 0.4711(u_2(5) - 0.082)$
$0.056 - 0.6155(u_2(6) - 0.051)$
$0.057 - 0.6622(u_3(7) - 0.165)$
$0.048 - 0.5601(u_3(8) - 0.11)$
$0.044 - 0.711(u_3(9) - 0.056)$
$J_o^* = 0.74186$
$J_o(\hat{\nu}) = 0.74212$

The above numerical computations clearly illustrate the convergence of J_o^* to $J_o(\hat{\nu})$ as $\varepsilon \to 0$.

5.8. Conclusions

In this chapter we have considered the average-cost-per-stage problem for finite-state Markov chains controlled by multiple decision makers. After formulating the general decision problem and obtaining certain fundamental existence results, we focused our attention on the multimodeling problem for a class of Markov models consisting of N weakly-coupled groups of strongly-interacting states. We have outlined a

procedure for obtaining near-optimal incentive policies, which allows the 'local' decision makers to use different simplified models of the system. Specifically, we have shown that each 'local' decision maker need only know the generator matrix of his own local Markov chain, the generator matrix describing the intergroup transitions, and the invariant measure of the other local chains along the optimal solution. Only the coordinator needs an exact knowledge of the 'global' model. The well-posedness of the procedure has been illustrated by a numerical example.

CHAPTER 6

INFORMATION INDUCED MULTIMODEL SOLUTIONS

6.1. Introduction

In the previous chapters we adopted a perturbational approach to the multimodeling problem. The crucial issue was one of well-posedness of the multimodel design. We had to establish the convergence of the optimal solution to the multimodel solution in the limit as the perturbational parameters go to zero.

In this chapter we attempt to induce a decomposition of the problem based on input-output considerations, such that the optimal solution within a class of admissible strategies, can be obtained from multiple reduced-order models with partial noninteraction among the decision makers.

In large scale systems, the DM's observe, in general, different variables through their individual objective functionals. These observed variables play a crucial role in the solution of the problem. Here we focus on the role of the observed variables in multimodel strategy design. We attempt to identify the core by examining the input structure and the observability structure induced by the observation sets of the DM's.

In Section 6.2 we formulate the problem, and discuss the structural decomposition and the class of admissible strategies referred to as Structure-Preserving strategies. In Section 6.3 we obtain multimodel solutions under FPS and FIS information patterns. In Section 6.4 we discuss decoupling of completely observable systems. In Section 6.5, we discuss

briefly extensions to many decision maker problems and Pareto games. In Section 6.6, we study applications of the concepts to control of large scale interconnected subsystems and multiarea power systems. Section 6.7 concludes the chapter.

6.2. Problem Formulation

6.2.1. The problem

Consider a linear system controlled by two DMs,

$$\dot{x} = Ax + B_1u_1 + B_2u_2 \; ; \quad x(0) = x_o \tag{6.1a}$$

$$y_i = C_ix; \quad i = 1,2 \tag{6.1b}$$

$$\dim x = n, \; \dim u_i = m_i, \quad \dim y_i = p_i$$

The variables y_i will be referred to as the 'observation set' of each DM. These are in fact the controlled variables as seen through the performance index of each DM, and may or may not correspond to the actual system outputs available to each DM.

The performance index of each DM is given by

$$J_i(\gamma_1,\gamma_2) = \{ \frac{1}{2} \int_0^\infty (y_i'y_i + u_i'R_iu_i)dt \mid u_i(t) = \gamma_i(\cdot)\}; \; i = 1,2 \tag{6.2}$$

where $\gamma_i(\cdot)$ is the admissible strategy of DMi, measurable with respect to the sigma-algebra generated by his information set (to be specified later).

The DMs are to select optimal strategies $\{\gamma_i^* | \gamma_i^* \in \Gamma_i;\ i = 1,2\}$ such that

$$J_i(\gamma_i^*,\gamma_j^*) \leq J_i(\gamma_i,\gamma_j^*); \quad \forall\ \gamma_i \in \Gamma_i; \quad i,j = 1,2; \quad i \neq j \tag{6.3}$$

where $\{\Gamma_i;\ i = 1,2\}$ are some admissible strategy sets for the DMs to be specified later. The pair of inequalities in (6.3) define the Nash equilibrium point.

In large scale game problems, the 'curse of dimensionality' may render any direct approach to the optimal solution computationally intractable. Hence there is a strong motivation for the DMs to look for alternative approaches to the problem which ease the computational difficulties. The approach formulated in the sequel has the desirable feature that it induces a partial noninteraction among the DMs leading to a lower order game. This is done by choosing appropriate admissible strategy sets Γ_i based on a particular structural decomposition of the system.

6.2.2. Structural decomposition

The observation sets of the DMs given by (6.1b) induce a certain observability decomposition on the state space. We propose to exploit this decomposition to obtain multimodel strategies. To do this, we start by exhibiting this observability decomposition explicitly by transforming the state space. This may be done either by performing chained aggregation sequentially with respect to each DM's observation set [8,54,55]; or, equivalently by making a similarity transformation directly, following a procedure dual to the one in [56,61] where a controllability decomposition was achieved.

The transformed system is represented as,

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}_1 u_1 + \bar{B}_2 u_2; \quad \bar{x}(0) = \bar{x}_o \tag{6.4a}$$

$$y_i = \bar{C}_i \bar{x} \; ; \quad i = 1,2 \tag{6.4b}$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & 0 & \bar{A}_{13} & 0 \\ 0 & \bar{A}_{22} & \bar{A}_{23} & 0 \\ 0 & 0 & \bar{A}_{33} & 0 \\ \bar{A}_{41} & \bar{A}_{42} & \bar{A}_{43} & \bar{A}_{44} \end{bmatrix}$$

$$\bar{C}_1 = [\, \bar{C}_{11} \quad 0 \quad \bar{C}_{13} \quad 0 \,]$$

$$\bar{C}_2 = [\, 0 \quad \bar{C}_{22} \quad \bar{C}_{23} \quad 0 \,]$$

$$\bar{B}_i = \begin{bmatrix} \bar{B}_{i1} \\ \bar{B}_{i2} \\ \bar{B}_{i3} \\ \bar{B}_{i4} \end{bmatrix} \; ; \quad i = 1,2 \quad ;$$

and

$$[\, \bar{A}_{ii}, \bar{C}_{ii} \,] \; , \; \left[\begin{bmatrix} \bar{A}_{ii} & \bar{A}_{i3} \\ 0 & \bar{A}_{33} \end{bmatrix} \; , \; [\, \bar{C}_{ii} \quad \bar{C}_{i3} \,] \right] ; \; i = 1,2$$

are observable pairs.

The eigenvalues of $\{\bar{A}_{ii};\ i = 1,2\}$ represent the modes which are observable only to DMi but not to DMj $(i \neq j)$; the eigenvalues of $\bar{A}_{33}$ represent the modes which are observable to both the DMs; and the eigenvalues of $\bar{A}_{44}$ represent the modes which are unobservable to both the DMs.

For simplicity we shall neglect the jointly unobservable modes. In a well formulated problem these modes are stable and do not contribute anything to the cost. Hence, from now onwards we shall assume the system matrices to have the following form:

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & 0 & \bar{A}_{13} \\ 0 & \bar{A}_{22} & \bar{A}_{23} \\ 0 & 0 & \bar{A}_{33} \end{bmatrix} \tag{6.5a}$$

$$\begin{aligned} \bar{C}_1 &= [\ \bar{C}_{11} \quad 0 \quad \bar{C}_{13}\ ] \\ \bar{C}_2 &= [\ 0 \quad \bar{C}_{22} \quad \bar{C}_{23}\ ] \end{aligned} \tag{6.5b}$$

$$\bar{B}_i = \begin{bmatrix} \bar{B}_{i1} \\ \bar{B}_{i2} \\ \bar{B}_{i3} \end{bmatrix} \quad ; \quad i = 1,2 \quad . \tag{6.5c}$$

The input structure specified by the matrices $\bar{B}_1$, $\bar{B}_2$ are not in a form suitable for our analysis. We need to make input space transformations

in order to appropriately overlap the input structure with the observability decomposition. Assuming that the pairs $\{(\bar{A},\bar{B}_i),\ (\bar{A}_{ii},\bar{B}_{ii});\ i = 1,2\}$ are controllable, there exist matrices G_1, G_2 such that the input space transformation $\{u_i = G_i\bar{u}_i;\ i = 1,2\}$ gives the new input matrices the following form [57].

$$\hat{B}_1 = \bar{B}_1G_1 = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{14} \\ 0 & \hat{B}_{12} \\ 0 & \hat{B}_{13} \end{bmatrix}, \qquad \hat{B}_2 = \bar{B}_2G_2 = \begin{bmatrix} 0 & \hat{B}_{21} \\ \hat{B}_{22} & \hat{B}_{24} \\ 0 & \hat{B}_{23} \end{bmatrix} \tag{6.6}$$

where the pairs $\{(\bar{A}_{ii},\hat{B}_{ii});\ i = 1,2\}$ are controllable.

<u>Remarks</u>: Before performing the input space transformation, we might need to do another state space transformation; but this can be done without destroying the observability decomposition. This is to put the system in an appropriate basis such that $\mathcal{X} = \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \mathcal{V}$ where $\mathcal{X}$ is the state space and $\mathcal{R}_i$ is a controllability subspace of DMi. The input space transformations G_i identify explicitly the control channels through which the individually observable modes are completely controllable [57].

6.2.3. Structure-Preserving strategies

The system and the performance indices, after the observability decomposition and the input space transformation, take the following form:

$$\dot{\bar{x}} = \bar{A}\bar{x} + \hat{B}_1\bar{u}_1 + \hat{B}_2\bar{u}_2\ ;\ \bar{x}(0) = \bar{x}_0 \tag{6.7a}$$

$$y_i = \bar{C}_i\bar{x}\ ;\quad i = 1,2 \tag{6.7b}$$

$$J_i = \frac{1}{2}\int_0^{\infty}(y_i'y_i + \bar{u}_i'\hat{R}_i\bar{u}_i)dt \ ; \ i = 1,2 \ ; \tag{6.8}$$

where $\hat{R}_i = G_i'R_iG_i$. We shall assume that

$$\hat{R}_i = \begin{bmatrix} \hat{R}_{ii} & 0 \\ 0 & \hat{R}_{ij} \end{bmatrix} > 0 \ ; \ \ i,j = 1,2 \ ; \ i \neq j \ .$$

The nature of the results obtained here hold for arbitrary positive definite $\hat{R}_i$; but assuming a block-diagonal form results in simpler derivations.

Before we obtain the Nash solution, we need to define the set of admissible strategies for each DM. The admissible strategy sets that we are particularly interested in here will be referred to as 'Structure-Preserving' strategies and are defined below.

Definition: A Structure-Preserving strategy set is the set of all linear feedback strategies which preserve the observability decomposition (6.5) of the closed-loop system.

In the single DM case, the three-component-control of [55,62] is a Structure-Preserving control. After the first component achieves decoupling, the second and third components which control the aggregate and the residual, respectively, are Structure-Preserving. The design in [55] was purely from a pole-placement point of view without any optimality considerations. Here we shall show that in the multiple DM case, the design of Structure-Preserving Nash strategies leads to multimodel solutions.

6.3 Multimodel Solutions

We shall consider two types of information patterns for the DMs: the Feedback Perfect State (FPS) and the Feedback Imperfect State (FIS). Under the FPS information pattern, each DM knows, at time t, the current state of the system, x(t); and under the FIS information pattern, each DM knows, at time t, only the current value of his observation, y(t).

6.3.1. FPS information pattern

Under the FPS information pattern, the admissible strategy set Γ_i of DMi is the set of linear state feedback strategies which are Structure-Preserving. Specifically,

$$\Gamma_i = \{ \gamma_i | \gamma_i(\bar{x}) = -F_i\bar{x} = -\begin{bmatrix} F_{i1}\delta_{i1} & F_{i1}\delta_{i2} & F_{i3} \\ 0 & 0 & F_{3i} \end{bmatrix} \bar{x} \}; \; i = 1,2 \; ; \tag{6.9}$$

where δ_{ij} is the Kronecher delta.

Now, to find the Nash solution, we need to find a pair $\{\gamma_i^* \in \Gamma_i \; ; \; i = 1,2\}$ such that the pair of inequalities (6.3) are satisfied. Substituting $\bar{u}_i = \gamma_i(\bar{x})$ from (6.9) in (6.7) and (6.8) we get

$$\dot{\bar{x}} = \hat{A}\bar{x} \; ; \quad \bar{x}(0) = \bar{x}_o \tag{6.10a}$$

$$y_i = \bar{C}_i\bar{x} \tag{6.10b}$$

$$J_i = \frac{1}{2}\int_0^\infty (\bar{x}'Q_i\bar{x})dt \; ; \quad i = 1,2 \; ; \tag{6.11}$$

where

$$Q_i = \bar{C}_i'\bar{C}_i + F_i'\hat{R}_iF_i \; ; \quad i = 1,2 \; ; \tag{6.12}$$

and the closed-loop system matrix is

$$\hat{A} = A - \hat{B}_1F_1 - \hat{B}_2F_2 = \begin{bmatrix} (\bar{A}_{11} - \hat{B}_{11}F_{11}) & 0 & (\bar{A}_{13} - \hat{B}_{11}F_{13} - \hat{B}_{14}F_{31} - \hat{B}_{21}F_{32}) \\ 0 & (\bar{A}_{22} - \hat{B}_{22}F_{22}) & (\bar{A}_{23} - \hat{B}_{22}F_{23} - \hat{B}_{24}F_{32} - \hat{B}_{12}F_{31}) \\ 0 & 0 & (\bar{A}_{33} - \hat{B}_{13}F_{13} - \hat{B}_{23}F_{32}) \end{bmatrix}$$

$$\triangleq \begin{bmatrix} \hat{A}_{11} & 0 & \hat{A}_{13} \\ 0 & \hat{A}_{22} & \hat{A}_{23} \\ 0 & 0 & \hat{A}_{33} \end{bmatrix} \quad . \tag{6.13}$$

The optimal solution $\{F^*_{ii}, F^*_{i3}, F^*_{3i} \; ; \; i = 1,2\}$ will depend in general on the initial conditions $\bar{x}_o$ [58]. To remove this dependence, we follow [58] and assume that the initial conditions are random with

$$E[\bar{x}_o\bar{x}_o'] = N > 0 \quad , \tag{6.14}$$

and modify the cost functionals to be

$$J_i = \frac{1}{2} \underset{\bar{x}_o}{E} \{ \int_0^\infty (\bar{x}'Q_i\bar{x})dt \} \; ; \quad i = 1,2 \quad . \tag{6.15}$$

Introduce M_1, M_2, $L \in \mathbb{R}^{n\times n}$ defined by

$$\frac{1}{2} \bar{x}_o'M_i\bar{x}_o = \frac{1}{2} \int_0^\infty (\bar{x}'Q_i\bar{x})dt; \quad i = 1,2 \tag{6.16}$$

$$L = \int_0^\infty E[\bar{x}(t)\bar{x}'(t)]dt \quad . \tag{6.17}$$

For any given pair (F_1, F_2) such that $\text{Re}\lambda(\hat{A}) < 0$, $M_i \geq 0$ and $L > 0$ satisfy the matrix Lyapunov equations

$$M_i\hat{A} + \hat{A}'M_i + Q_i = 0 \; ; \quad i = 1,2 \tag{6.18}$$

$$\hat{A}L + L\hat{A}' + N = 0 \; . \tag{6.19}$$

Partition M_i, L, N appropriately

$$M_i = \begin{bmatrix} M_{11}^{(i)} & M_{12}^{(i)} & M_{13}^{(i)} \\ M_{12}^{(i)'} & M_{22}^{(i)} & M_{23}^{(i)} \\ M_{13}^{(i)'} & M_{23}^{(i)'} & M_{33}^{(i)} \end{bmatrix} \; ; \quad i = 1,2 \tag{6.20a}$$

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12}' & L_{22} & L_{23} \\ L_{13}' & L_{23}' & L_{33} \end{bmatrix} \; ; \quad N = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{12}' & N_{22} & N_{23} \\ N_{13}' & N_{23}' & N_{33} \end{bmatrix} \tag{6.20b}$$

Applying the Matrix Minimum Principle [59], the optimal F_{ii}^*, F_{i3}^*, F_{3i}^* for the Feedback Nash solution* can be shown to satisfy (for i,j = 1,2; i ≠ j)

$$\hat{R}_{ii}F_{ii}^*L_{ii} + \hat{R}_{ii}F_{i3}^*L_{i3}' - \hat{B}_{ii}'M_{ii}^{(i)}L_{ii} - \hat{B}_{ii}'M_{i3}^{(i)}L_{i3}' = 0 \tag{6.21a}$$

$$\hat{R}_{ii}F_{ii}^*L_{i3} + \hat{R}_{ii}F_{i3}^*L_{33} - \hat{B}_{ii}'M_{i3}^{(i)}L_{33} - \hat{B}_{ii}'M_{ii}^{(i)}L_{i3} = 0 \tag{6.21b}$$

$$\hat{R}_{ij}F_{3i}^*L_{33} - \hat{B}_{i3}'M_{i3}^{(i)'}L_{i3} - \hat{B}_{i3}'M_{33}^{(i)}L_{33} - \hat{B}_{i4}'M_{ii}^{(i)}L_{i3} - \hat{B}_{i4}'M_{i3}^{(i)}L_{33} = 0 \tag{6.21c}$$

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*By an abuse of terminology, we shall refer to the solution as Feedback Nash. It is not Feedback Nash in the sense defined in [9] because it does not satisfy the Principle of Optimality. It is Nash in feedback information pattern.

where

$$M_{ij}^{(i)} = M_{jj}^{(i)} = M_{j3}^{(i)} = 0 \tag{6.22a}$$

$$M_{ii}^{(i)}\hat{A}_{ii}^{*} + \hat{A}_{ii}^{*\prime}M_{ii}^{(i)} + \bar{C}_{ii}^{\prime}\bar{C}_{ii} + F_{ii}^{*\prime}\hat{R}_{ii}F_{ii}^{*} = 0 \tag{6.22b}$$

$$M_{ii}^{(i)}\hat{A}_{i3}^{*} + M_{i3}^{(i)}\hat{A}_{33}^{*} + \hat{A}_{ii}^{*\prime}M_{i3}^{(i)} + \bar{C}_{ii}^{\prime}\bar{C}_{i3} + F_{ii}^{*\prime}\hat{R}_{ii}F_{i3}^{*} = 0 \tag{6.22c}$$

$$\begin{aligned} M_{33}^{(i)}\hat{A}_{33}^{*} + \hat{A}_{33}^{*\prime}M_{33}^{(i)} + M_{i3}^{(i)\prime}\hat{A}_{i3}^{*} + \hat{A}_{i3}^{*\prime}M_{i3} + \bar{C}_{i3}^{\prime}\bar{C}_{i3} & \\ + F_{i3}^{*\prime}\hat{R}_{ii}F_{i3}^{*} + F_{3i}^{*\prime}\hat{R}_{ij}F_{3i}^{*} &= 0 \end{aligned} \tag{6.22d}$$

$$\hat{A}_{ii}^{*}L_{i3} + \hat{A}_{i3}^{*}L_{33} + L_{i3}\hat{A}_{33}^{*\prime} + N_{i3} = 0 \tag{6.23a}$$

$$\hat{A}_{33}^{*}L_{33} + L_{33}\hat{A}_{33}^{*\prime} + N_{33} = 0 \tag{6.23b}$$

$\{\hat{A}_{ii}^{*}, \hat{A}_{i3}^{*}, \hat{A}_{33}^{*} ; i = 1,2\}$ are as in (6.13) with $\{F_{ii} = F_{ii}^{*}, F_{i3} = F_{i3}^{*}, F_{3i} = F_{3i}^{*}; i = 1,2\}$. Solving (6.21) we obtain,

$$F_{ii}^{*} = \hat{R}_{ii}^{-1}\hat{B}_{ii}^{\prime}M_{ii}^{(i)} \tag{6.24a}$$

$$F_{i3}^{*} = \hat{R}_{ii}^{-1}\hat{B}_{ii}^{\prime}M_{i3}^{(i)} \tag{6.24b}$$

$$F_{3i}^{*} = \tilde{R}_{ij}^{-1}\hat{B}_{i3}^{\prime}[M_{33}^{(i)} + M_{i3}^{(i)\prime}L_{i3}L_{33}^{-1}] + \hat{R}_{ij}^{-1}\hat{B}_{i4}^{\prime}[M_{i3}^{(i)} + M_{ii}^{(i)}L_{i3}L_{33}^{-1}] \tag{6.24c}$$

Notice that even though equations (6.21a) and (6.21b) are coupled in F_{ii}^{*} and F_{i3}^{*}, we are able to solve for them explicitly as in (6.24a) and (6.24b). This fact plays a crucial role in showing that the Nash solution admits a partial noninteraction. Substituting (6.24a) in (6.22b) we obtain,

$$M_{ii}^{(i)}\bar{A}_{ii} + \bar{A}_{ii}'M_{ii}^{(i)} + \bar{C}_{ii}'\bar{C}_{ii} - M_{ii}^{(i)}\hat{B}_{ii}\hat{R}_{ii}^{-1}\hat{B}_{ii}'M_{ii}^{(i)} = 0 \; ; \; i = 1,2 \qquad (6.25)$$

It can be readily seen from (6.24a) and (6.25) that F_{ii}^* is the solution of an optimal state regulator problem with parameters $(\bar{A}_{ii}, \hat{B}_{ii}, \bar{C}_{ii}, \hat{R}_{ii})$. The following proposition highlights the multimodel nature of the Nash solution.

Proposition 6.1:

Given the linear system (6.7) controlled by two DMs, and their performance indices (6.8), the design of Structure-Preserving Feedback Nash strategies under the FPS information pattern, leads to two low-order coupled optimization problems defined by

$$\min_{\bar{u}_i} \; J_i = \underset{z_{io}}{E} \left\{ \frac{1}{2}\int_0^\infty (y_i'y_i + \bar{u}_i'\hat{R}_i\bar{u}_i)dt \right\}$$

subject to

$$\bar{u}_i = \gamma_i(z_i) = -\begin{bmatrix} F_{ii} & F_{i3} \\ 0 & F_{3i} \end{bmatrix} z_i \; .$$

where

$$\dot{z}_i = \begin{bmatrix} \bar{A}_{ii} & \bar{A}_{i3} - \hat{B}_{ji}F_{3j} \\ 0 & \bar{A}_{33} - \hat{B}_{j3}F_{3j} \end{bmatrix} z_i + \begin{bmatrix} \hat{B}_{ii} & \hat{B}_{i4} \\ 0 & \hat{B}_{i3} \end{bmatrix} \bar{u}_i \; ; \; z_i(0) = z_{io}$$

$$y_i = [\bar{C}_{ii} \quad \bar{C}_{i3}] z_i$$

$$E[z_{io}\ z_{io}'] = \begin{bmatrix} N_{ii} & N_{i3} \\ N_{i3}' & N_{33} \end{bmatrix}$$

$$i,j = 1,2\ ;\quad i \neq j\quad .$$

The solution to this pair of coupled optimization problems admits partial noninteraction among the DMs, and is given by the set of equations (6.22)-(6.25).

At this point we would like to remark that the controllability-observability of the triple $\{(\bar{A}_{ii}, \hat{B}_{ii}, \bar{C}_{ii});\ i = 1,2\}$ guarantees

$$\mathrm{Re}\ \lambda(\hat{A}_{ii}^{*}) < 0\ ;\ i = 1,2 \tag{6.26}$$

For the solution to be well-defined we need only to verify that $\mathrm{Re}\ \lambda(\hat{A}_{33}^{*}) < 0$.

The coupling between the optimization problems of the two DMs is due to the presence of the control gain F_{3j} of DMj in DMi's low-order model. Partial noninteraction is achieved because each DM can evaluate his control gain F_{ii}^{*} independently in a decentralized manner by solving equations (6.24a) and (6.25). The control gains $\{F_{i3}^{*}, F_{3i}^{*};\ i = 1,2\}$ are then obtained by solving the coupled set of equations (6.22c-d), (6.23), (6.24).

Hence, we have succeeded in identifying the 'core' of a high-order game problem where the DMs actually interact, and a pair of low-order control problems, one for each DM. This has been achieved by restricting the admis-

sible strategy sets of the DMs to Structure-Preserving strategies under the FPS information pattern; and transforming the state space and input space appropriately.

Notice that F^*_{ii} is independent of the statistics of the initial conditions since it is obtained from (6.24a) and (6.25). But $\{F^*_{i3}, F^*_{3i}; i = 1,2\}$ do depend, in general, on the statistics of the initial conditions, as they are obtained from the coupled set of equations (6.22c-d), (6.23), (6.24), which may be difficult to solve in practice. The gain matrices $\{F^*_{i3}, F^*_{3i}; i = 1,2\}$ which result in $\{L_{i3} = 0; i = 1,2\}$ are of particular interest, as they are computationally simpler to obtain. Such a set is given by

$$F^*_{i3} = \hat{R}^{-1}_{ii}\hat{B}'_{ii}M^{(i)}_{i3} \; ; \quad i = 1,2 \tag{6.27a}$$

$$F^*_{3i} = \hat{R}^{-1}_{ij}[\hat{B}'_{i3}M^{(i)}_{33} + \hat{B}'_{i4}M^{(i)}_{i3}] \; ; \; i,j = 1,2 \; ; \quad i \neq j \tag{6.27b}$$

Here $M^{(i)}_{i3}$, $M^{(i)}_{33}$ satisfy the coupled set of equations (for $i,j = 1,2$; $i \neq j$),

$$\begin{aligned} & M^{(i)}_{ii}\bar{A}_{i3} + M^{(i)}_{i3}\bar{A}_{33} + \bar{A}'_{ii}M^{(i)}_{i3} + \bar{C}'_{ii}\bar{C}_{i3} - M^{(i)}_{ii}S_{ii}M^{(i)}_{i3} \\ & - M^{(i)}_{ii}\hat{S}_i M^{(i)}_{33} - M^{(i)}_{ii}S_{i4}M^{(i)}_{i3} - M^{(i)}_{ii}\tilde{S}_{ji}M^{(j)}_{33} - M^{(i)}_{ii}\bar{S}_{ji}M^{(j)}_{j3} \\ & - M^{(i)}_{i3}S_{i3}M^{(i)}_{33} - M^{(i)}_{i3}\hat{S}'_i M^{(i)}_{i3} - M^{(i)}_{i3}S_{j3}M^{(j)}_{33} - M^{(i)}_{i3}\hat{S}'_j M^{(j)}_{j3} = 0 \end{aligned} \tag{6.28a}$$

$$M_{33}^{(i)}\bar{A}_{33} + \bar{A}_{33}' M_{33}^{(i)} + M_{i3}^{(i)'}\bar{A}_{i3} + \bar{A}_{i3}' M_{i3}^{(i)} + \bar{C}_{i3}'\bar{C}_{i3} - M_{33}^{(i)} S_{i3} M_{33}^{(i)} - M_{33}^{(i)}\hat{S}_i' M_{i3}^{(i)}$$

$$- M_{i3}^{(i)'}\hat{S}_i M_{33}^{(i)} - M_{33}^{(i)} S_{j3} M_{33}^{(j)} - M_{33}^{(j)} S_{j3} M_{33}^{(i)} - M_{33}^{(i)}\hat{S}_j' M_{j3}^{(j)} - M_{j3}^{(j)'}\hat{S}_j M_{33}^{(i)}$$

$$- M_{i3}^{(i)'} S_{ii} M_{i3}^{(i)} - M_{i3}^{(i)} S_{i4} M_{i3}^{(i)} - M_{i3}^{(i)'}\tilde{S}_{ji} M_{33}^{(j)}$$

$$- M_{33}^{(j)}\tilde{S}_{ji}' M_{i3}^{(i)} - M_{i3}^{(i)}\bar{S}_{ji} M_{j3}^{(j)} - M_{j3}^{(j)'}\bar{S}_{ji}' M_{i3}^{(i)} = 0 \qquad (6.28b)$$

where,

$$S_{ii} = \hat{B}_{ii}\hat{R}_{ii}^{-1}\hat{B}_{ii}'; \quad S_{i3} = \hat{B}_{i3}\hat{R}_{ij}^{-1}\hat{B}_{i3}'; \quad \tilde{S}_{ij} = \hat{B}_{ij}\hat{R}_{ij}^{-1}\hat{B}_{i3}'$$

$$S_{i4} = \hat{B}_{i4}\hat{R}_{ij}^{-1}\hat{B}_{i4}'; \quad \bar{S}_{ij} = \hat{B}_{ij}\hat{R}_{ij}^{-1}\hat{B}_{i4}'; \quad \hat{S}_i = \hat{B}_{i4}\hat{R}_{ij}^{-1}\hat{B}_{i3}'$$

Furthermore $\{F_{i3}^*, F_{3i}^*;\ i = 1,2\}$ are such that

$$\hat{A}_{i3}^* L_{33} + N_{i3} = 0\ ;\quad i = 1,2\ ; \qquad (6.29)$$

where L_{33} is the positive definite solution of (6.23b).

Notice that if the initial cross-covariance $N_{13} = N_{23} = 0$, then (6.29) is satisfied if and only if $\hat{A}_{13}^* = \hat{A}_{23}^* = 0$; which would be true if the solution of (6.27) and (6.28) block-diagonalizes the closed-loop system.

6.3.2. FIS information pattern

It can be readily seen that when the output matrices are of the form given by (6.5b) Structure-Preserving strategies involving only static linear output feedback do not exist.

When the output matrices split so that there are separate observation channels for the individually and commonly observable modes, i.e., when

$$\bar{C}_i = \begin{bmatrix} \bar{C}_{ii}\delta_{i1} & \bar{C}_{ii}\delta_{i2} & 0 \\ 0 & 0 & \bar{C}_{i3} \end{bmatrix} ; \quad i = 1,2 \quad ; \tag{6.30}$$

linear static output feedback Structure-Preserving strategies do exist, and belong to the admissible strategy set $\tilde{\Gamma}_i$ defined by,

$$\tilde{\Gamma}_i = \{ \tilde{\gamma}_i | \tilde{\gamma}_i(y_i) = - \tilde{F}_i y_i = - \begin{bmatrix} \tilde{F}_{ii} & \tilde{F}_{i3} \\ 0 & \tilde{F}_{3i} \end{bmatrix} y_i \}; \ i = 1,2 \ . \tag{6.31}$$

Substituting $\bar{u}_i = \tilde{\gamma}_i(y_i)$ from (6.31) in (6.7) and (6.8), we get

$$\dot{\bar{x}} = \tilde{A}\bar{x} \ ; \ \ \bar{x}(0) = \bar{x}_o \tag{6.32a}$$

$$y_i = \bar{C}_i\bar{x} \tag{6.32b}$$

$$J_i = \frac{1}{2}\int_0^\infty (\bar{x}'\tilde{Q}_i\bar{x})dt; \quad i = 1,2 \quad ; \tag{6.33}$$

where

$$\tilde{Q}_i = \bar{C}_i'(I + \tilde{F}_i'\hat{R}_i\tilde{F}_i)\bar{C}_i \ ; \quad i = 1,2 \quad ; \tag{6.34}$$

and the closed-loop system matrix becomes ,

$$\tilde{A} = \bar{A} - \hat{B}_1\tilde{F}_1\bar{C}_1 - \hat{B}_2\tilde{F}_2\bar{C}_2$$

$$= \begin{bmatrix} (\bar{A}_{11} - \hat{B}_{11}\tilde{F}_{11}\bar{C}_{11}) & 0 & (\bar{A}_{13} - \hat{B}_{13}\tilde{F}_{13}\bar{C}_{13} - \hat{B}_{14}\tilde{F}_{31}\bar{C}_{13} - \hat{B}_{21}\tilde{F}_{32}\bar{C}_{23}) \\ 0 & (\bar{A}_{22} - \hat{B}_{22}\tilde{F}_{22}\bar{C}_{22}) & (\bar{A}_{23} - \hat{B}_{22}\tilde{F}_{23}\bar{C}_{23} - \hat{B}_{24}\tilde{F}_{32}\bar{C}_{23} - \hat{B}_{12}\tilde{F}_{31}\bar{C}_{13}) \\ 0 & 0 & (\bar{A}_{33} - \hat{B}_{13}\tilde{F}_{31}\bar{C}_{13} - \hat{B}_{23}\tilde{F}_{32}\bar{C}_{23}) \end{bmatrix}$$

$$\triangleq \begin{bmatrix} \tilde{A}_{11} & 0 & \tilde{A}_{13} \\ 0 & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & 0 & \tilde{A}_{33} \end{bmatrix} \quad . \tag{6.35}$$

Define M_1, M_2 and L as in (6.16), (6.17), and partition them as in (6.20). For any given pair $(\tilde{F}_1,\tilde{F}_2)$, such that Re $\lambda(\tilde{A}) < 0$, $M_i \geq 0$ and $L > 0$ satisfy the matrix Lyapunov equations

$$M_i\tilde{A} + \tilde{A}'M_i + \tilde{Q}_i = 0 \; ; \; i = 1,2 \quad . \tag{6.36}$$

$$\tilde{A}L + L\tilde{A}' + N = 0 \quad . \tag{6.37}$$

Applying the Matrix Minimum Principle [59], the optimal $\tilde{F}^*_{ii}$, $\tilde{F}^*_{13}$, $\tilde{F}^*_{3i}$ for the Feedback Nash solution can be shown to satisfy (for $i,j = 1,2$; $i \neq j$)

$$\tilde{F}^*_{ii}\bar{C}_{ii}L_{ii}\bar{C}'_{ii} + \tilde{F}^*_{i3}\bar{C}_{i3}L'_{i3}\bar{C}'_{ii} = \hat{R}^{-1}_{ii}\hat{B}'_{ii}[M^{(i)}_{ii}L_{ii}\bar{C}'_{ii} + M^{(i)}_{i3}L'_{i3}\bar{C}'_{ii}] \quad (6.38a)$$

$$\tilde{F}^*_{ii}\bar{C}_{ii}L_{i3}\bar{C}'_{i3} + \tilde{F}^*_{i3}\bar{C}_{i3}L_{33}\bar{C}'_{i3} = \hat{R}^{-1}_{ii}\hat{B}'_{ii}[M^{(i)}_{ii}L_{i3}\bar{C}'_{i3} + M^{(i)}_{i3}L_{33}\bar{C}'_{i3}] \quad (6.38b)$$

$$\tilde{F}^*_{3i} = \hat{R}^{-1}_{ij}[\hat{B}'_{i3}M^{(i)}_{33}L_{33}\bar{C}'_{i3} + \hat{B}'_{i3}M^{(i)'}_{i3}L_{i3}\bar{C}'_{i3} + \hat{B}'_{i4}M^{(i)}_{ii}L_{i3}\bar{C}'_{i3} + \hat{B}'_{i4}M^{(i)}_{i3}L_{33}\bar{C}'_{i3}]$$

$$\times\ [\bar{C}_{i3}L_{33}\bar{C}'_{i3}]^{-1} \quad (6.38c)$$

where

$$M^{(i)}_{ij} = M^{(i)}_{jj} = M^{(i)}_{j3} = 0 \quad (6.39a)$$

$$M^{(i)}_{ii}\tilde{A}^*_{ii} + \tilde{A}^{*'}_{ii}M^{(i)}_{ii} + \bar{C}'_{ii}(I + \tilde{F}^{*'}_{ii}\hat{R}_{ii}\tilde{F}^*_{ii})\bar{C}_{ii} = 0 \quad (6.39b)$$

$$M^{(i)}_{ii}\tilde{A}^*_{i3} + M^{(i)}_{i3}\tilde{A}^*_{33} + \tilde{A}^{*'}_{ii}M^{(i)}_{i3} + \bar{C}'_{ii}(I + \tilde{F}^{*'}_{ii}\hat{R}_{ii}\tilde{F}^*_{i3})\bar{C}_{i3} = 0 \quad (6.39c)$$

$$M^{(i)}_{33}\tilde{A}^*_{33} + \tilde{A}^{*'}_{33}M^{(i)}_{33} + M^{(i)}_{i3}\tilde{A}^*_{i3} + \tilde{A}^{*'}_{i3}M^{(i)}_{i3} + \bar{C}'_{i3}(I + \tilde{F}^{*'}_{i3}\hat{R}_{ii}\tilde{F}^*_{i3}$$

$$+ \tilde{F}^{*'}_{3i}\hat{R}_{ij}\tilde{F}^*_{3i})\bar{C}_{i3} = 0 \quad (6.39d)$$

$$\tilde{A}^*_{ii}L_{ii} + L_{ii}\tilde{A}^{*'}_{ii} + \tilde{A}^*_{i3}L'_{i3} + L_{i3}\tilde{A}^{*'}_{i3} + N_{ii} = 0 \quad (6.40a)$$

$$\tilde{A}^*_{ii}L_{i3} + \tilde{A}^*_{i3}L_{33} + L_{i3}\tilde{A}^{*'}_{33} + N_{i3} = 0 \quad (6.40b)$$

$$\tilde{A}^*_{33}L_{33} + L_{33}\tilde{A}^{*'}_{33} + N_{33} = 0 \quad (6.40c)$$

$\{\tilde{A}^*_{ii}, \tilde{A}^*_{i3}, \tilde{A}^*_{33}\ ;\ i = 1,2\}$ are as in (6.35) with $\{\tilde{F}_{ii} = \tilde{F}^*_{ii}, \tilde{F}_{i3} = \tilde{F}^*_{i3}, \quad \tilde{F}_{3i} = \tilde{F}^*_{3i}\ ;\ \ i = 1,2\}$. The following proposition highlights the multimodel nature of the Nash solution.

<u>Proposition 6.2:</u>

Given the linear system (6.7a) controlled by two DMs, their observation sets (6.30), and their performance indices (6.8), the design of Structure-Preserving Feedback Nash strategies under the FIS information pattern leads to two low-order coupled optimization problems defined by

$$\min_{\bar{u}_i} J_i = \underset{z_{io}}{E}\ \{ \frac{1}{2}\int_0^\infty (y_i'y_i + \bar{u}_i'\hat{R}_i\bar{u}_i)dt \}$$

subject to

$$\bar{u}_i = \tilde{\gamma}_i(y_i) = - \begin{bmatrix} \tilde{F}_{ii} & \tilde{F}_{i3} \\ 0 & \tilde{F}_{3i} \end{bmatrix} y_i$$

where

$$y_i = \begin{bmatrix} \bar{C}_{ii} & 0 \\ 0 & \bar{C}_{i3} \end{bmatrix} z_i$$

$$\dot{z}_i = \begin{bmatrix} \bar{A}_{ii} & \bar{A}_{i3} - \hat{B}_{ji}\tilde{F}_{3j}\bar{C}_{j3} \\ 0 & \bar{A}_{33} - \hat{B}_{j3}\tilde{F}_{3j}\bar{C}_{j3} \end{bmatrix} z_i + \begin{bmatrix} \hat{B}_{ii} & \hat{B}_{i4} \\ 0 & \hat{B}_{i3} \end{bmatrix} \bar{u}_i;\ z_i(0) = z_{io}$$

$$E[z_{io}\ \ z_{io}'] = \begin{bmatrix} N_{ii} & N_{i3} \\ N_{i3}' & N_{33} \end{bmatrix}$$

$i,j = 1,2;\ \ i \neq j$

The solution to this pair of coupled optimization problems is given by the set of equations (6.38)-(6.40).

Now, unlike the earlier problem of Section (6.31), we need to verify that

$$\text{Re}\ \lambda(\hat{A}^*_{ii}) < 0\ ;\ \ i = 1,2,3$$

for the solution to be well-defined. Also unlike the earlier case, the Structure-Preserving Feedback Nash solution of Proposition 6.2 is completely interacting. This is essentially due to the fact that equations (6.38a) and (6.38b) cannot be solved explicitly for $\tilde{F}^*_{ii}$ and $\tilde{F}^*_{i3}$. Another significant difference is that now all the optimal gains $\{F^*_{ii}, F^*_{i3}, F^*_{3i};\ i = 1,2\}$ depend on the statistics of the initial conditions.

Partial noninteraction results when $\{L_{i3} = 0;\ i = 1,2\}$. In this case the optimal solution is given by $(i,j = 1,2;\ i \neq j)$

$$\tilde{F}^*_{ii} = \hat{R}^{-1}_{ii}\hat{B}'_{ii}M^{(i)}_{ii}L_{ii}\bar{C}'_{ii}(\bar{C}_{ii}L_{ii}\bar{C}'_{ii})^{-1} \tag{6.41a}$$

$$\tilde{F}^*_{i3} = \hat{R}^{-1}_{ii}\hat{B}'_{ii}M^{(i)}_{i3}L_{33}\bar{C}'_{i3}(\bar{C}_{i3}L_{33}\bar{C}'_{i3})^{-1} \tag{6.41b}$$

$$\tilde{F}^*_{3i} = \hat{R}^{-1}_{ij}[\hat{B}'_{i3}M^{(i)}_{33}L_{33}\bar{C}'_{i3} + \hat{B}'_{i4}M^{(i)}_{i3}L_{33}\bar{C}'_{i3}](\bar{C}_{i3}L_{33}\bar{C}'_{i3})^{-1} \tag{6.41c}$$

$M_{ii}^{(i)}$, $M_{i3}^{(i)}$, $M_{33}^{(i)}$ are obtained from (6.39) with the control gains given by (6.41) L_{33} is obtained from (6.40c), and L_{ii} is the positive definite solution of

$$\tilde{A}_{ii}^{*}L_{ii} + L_{ii}\tilde{A}_{ii}^{*\prime} + N_{ii} = 0 \tag{6.42}$$

Furthermore $\{\tilde{F}_{i3}^{*}, \tilde{F}_{3i}^{*};\ i = 1,2\}$ are such that

$$\tilde{A}_{i3}^{*}L_{33} + N_{i3} = 0\ ;\quad i = 1,2 \tag{6.43}$$

Now $\tilde{F}_{ii}^{*}$ is first obtained by each DM independently on solving equations (6.39b), (6.41a) and (6.42). This is the optimal solution of an output regulator problem with parameters $(\bar{A}_{ii},\hat{B}_{ii},\bar{C}_{ii},\hat{R}_{ii},N_{ii})$ [58].

In cases when the output matrices do not split as in (6.30), the FPS Structure-Preserving Nash strategies of Proposition 6.1 can be synthesized as feedback strategies using dynamic observers.

We let,

$$\bar{u}_i = \bar{\gamma}_i(\hat{z}_i) = -\begin{bmatrix} F_{ii}^{*} & F_{i3}^{*} \\ 0 & F_{3i}^{*} \end{bmatrix}\hat{z}_i;\quad i = 1,2 \tag{6.44}$$

where,

$$\dot{\hat{z}}_i = \begin{bmatrix} \bar{A}_{ii} & \bar{A}_{i3} - \hat{B}_{ji}F_{3j}^{*} \\ 0 & \bar{A}_{33} - \hat{B}_{j3}F_{3j}^{*} \end{bmatrix}\hat{z}_i + \begin{bmatrix} \hat{B}_{ii} & \hat{B}_{i4} \\ 0 & \hat{B}_{i3} \end{bmatrix}\bar{u}_i + K_i\{y_i - [\bar{C}_{ii}\ \ \bar{C}_{i3}]\hat{z}_i\}$$

$$i,j = 1,2;\ i \neq j \tag{6.45}$$

$\{F^*_{ii}, F^*_{i3}, F^*_{3i};\ i = 1,2\}$ are given by equations (6.24), and K_i is the observer gain to be chosen by each DM.

Notice that the dimension of the observer of each DM is equal to the dimension of his own observable eigenspace, which is all be needs to reconstruct in order to implement the FPS Structure-Preserving strategy.

Defining $e_i = z_i - \hat{z}_i$, we get,

$$\dot{e}_i = \begin{bmatrix} \bar{A}_{ii} - K_{i1}\bar{C}_{ii} & \bar{A}_{i3} - \hat{B}_{ji}F^*_{3j} - K_{i1}\bar{C}_{i3} \\ -K_{i2}\bar{C}_{ii} & \bar{A}_{33} - \hat{B}_{j3}F^*_{3j} - K_{i2}\bar{C}_{i3} \end{bmatrix} e_i \ ; \quad K_i = \begin{bmatrix} K_{i1} \\ K_{i2} \end{bmatrix}$$

$$= \hat{A}_i e_i \ ; \quad i,j = 1,2 \ ; \quad i \neq j \tag{6.46}$$

If we choose K_i such that

$$\mathrm{Re}\,\lambda(\hat{A}_i) < -\frac{1}{\mu_i} ; \quad \mu_i > 0 \ ; \ i = 1,2 \tag{6.47}$$

then we can write,

$$\mu_i \dot{e}_i = \tilde{A}_i e_i$$

$$\mathrm{Re}\,\lambda(\tilde{A}_i) < -1 \ ; \quad i = 1,2 \tag{6.48}$$

Hence, by making the observer dynamics arbitrarily fast, we can represent the error system as a stable singularly perturbed system i.e.; $e_i \to 0$ as $\mu_i \to 0$.

Rewriting the composite system and the feedback strategies as,

$$\dot{\bar{x}} = \bar{A}\bar{x} + \hat{B}_1\bar{u}_1 + \hat{B}_2\bar{u}_2$$
$$\mu_i\dot{e}_i = \tilde{A}_i e_i \; ; \; i = 1,2 . \tag{6.49}$$

$$\bar{u}_i = \gamma_i(\bar{x},e_i) = -\begin{bmatrix} F^*_{i1}\delta_{i1} & F^*_{i1}\delta_{i2} & F^*_{i3} \\ 0 & 0 & F^*_{3i} \end{bmatrix}\bar{x} + \begin{bmatrix} F^*_{i1} & F^*_{i3} \\ 0 & F^*_{3i} \end{bmatrix} e_i ; \; i = 1,2 . \tag{6.50}$$

Since $e_i \to 0$ as $\mu_i \to 0$, $\gamma_i(\bar{x},e_i)$ converges in open-loop to a policy having a unique feedback representation, which we denote by $\bar{\gamma}^f_i(\bar{x})$; and

$$\bar{\gamma}^f_i(\bar{x}) = \gamma^*_i(\bar{x}); \quad i = 1,2 \tag{6.51}$$

where $\{\gamma^*_i(\bar{x}); \; i = 1,2\}$ is the FPS Structure-Preserving Feedback Nash strategy of Proposition 6.1.

Due to (6.51) and the results of [25], we have

$$\lim_{\|\mu\| \to 0} J_i(\gamma_1,\gamma_2) = J_i(\gamma^*_1,\gamma^*_2) \; ; \quad i = 1,2 . \tag{6.52}$$

It is to be noted that (6.50) is not the Feedback Nash strategy for the system (6.49) and the performance indices (6.8) within the class of admissible strategies $\hat{\Gamma}_i$ defined by

$$\Gamma_i = \{ \gamma_i \mid \gamma_i(\bar{x},e_i) = -\begin{bmatrix} F_{i1}\delta_{i1} & F_{i1}\delta_{i2} & F_{i3} \\ 0 & 0 & F_{3i} \end{bmatrix}\bar{x} + \begin{bmatrix} F_{i1} & F_{i3} \\ 0 & F_{3i} \end{bmatrix} e_i$$
$$= -\begin{bmatrix} F_{i1} & F_{i3} \\ 0 & F_{3i} \end{bmatrix} \hat{z}_i ; \quad K_i \text{ fixed } \} \; ; \; i = 1,2 \tag{6.53}$$

The Feedback Nash strategy $\hat{\gamma}_i^* \in \hat{\Gamma}_i$ will in general depend on the choice of the observer gains K_i. We do not compute $\hat{\gamma}_i^*$ because the strategy $\bar{\gamma}_i$ given by (6.44), or equivalently by (6.50), has the property of being near-equilibrium and a symptotic Nash [20] as established by the following proposition.

Proposition 6.3:

The strategy $\bar{\gamma}_i(\hat{z}_i) = \gamma_i(\bar{x}, e_i)$ given by (6.44) (or (6.50)) is near-equilibrium and asymptotic Nash within the class $\hat{\Gamma}_i$ defined by (6.53). That is,

$$\lim_{\|\mu\| \to 0} \{J_i(\bar{\gamma}_i, \bar{\gamma}_j) - J_i(\hat{\gamma}_i, \bar{\gamma}_j)\} = 0; \quad \forall \, \hat{\gamma}_i \in \hat{\Gamma}_i; \; i,j = 1,2; \; i \neq j$$

and,

$$\lim_{\|\mu\| \to 0} \{J_i(\bar{\gamma}_i, \hat{\gamma}_j) - J_i(\bar{\gamma}_i, \bar{\gamma}_j)\} = 0 \; ;$$

$\forall \, \hat{\gamma}_j \in \hat{\Gamma}_j$ such that $J_j(\bar{\gamma}_i, \hat{\gamma}_j) \leq J_j(\bar{\gamma}_i, \bar{\gamma}_j)$; $i,j = 1,2$; $i \neq j$.

The proof of the above Proposition follows readily from the results of Chapter 2.

6.4. Decoupling of Completely Observable Systems

In situations when the whole system is completely observable through the observation set of each DM, the 'core' is the full problem itself. But in some such cases, if the DMs have access to all the states then the observability decomposition can be induced by using state feedback. The role of the decoupling control in reduced-order modeling has been studied in detail in [62]. Here we shall outline the procedure for multiple DM problems.

Suppose after appropriate state space, input space and output space transformations, the system can be put in the following form [62],

$$\dot{\bar{x}} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} \\ \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} \end{bmatrix} \bar{x} + \begin{bmatrix} \bar{B}_{11} & \bar{B}_{14} \\ 0 & \bar{B}_{12} \\ 0 & \bar{B}_{13} \end{bmatrix} \bar{u}_1 + \begin{bmatrix} 0 & \bar{B}_{21} \\ \bar{B}_{22} & \bar{B}_{24} \\ 0 & \bar{B}_{23} \end{bmatrix} \bar{u}_2 \tag{6.54a}$$

$$\begin{aligned} \bar{y}_1 &= [\ \bar{C}_{11} \quad 0 \quad \bar{C}_{13}\ ]\,\bar{x} \\ \bar{y}_2 &= [\ 0 \quad \bar{C}_{22} \quad \bar{C}_{23}\ ]\,\bar{x} \end{aligned} \tag{6.54b}$$

with $\{\bar{B}_{i1}, \bar{B}_{i3},\ i = 1,2\}$ being square and nonsingular.

Now if the DMs use the following strategies,

$$\bar{u}_1 = \begin{bmatrix} 0 & -\bar{B}_{11}^{-1}(\bar{A}_{12} - \bar{B}_{21}\bar{B}_{23}^{-1}\bar{A}_{32}) & 0 \\ -\bar{B}_{13}^{-1}\bar{A}_{31} & 0 & 0 \end{bmatrix} \bar{x} + \hat{u}_1 \tag{6.55a}$$

$$\bar{u}_2 = \begin{bmatrix} -\bar{B}_{22}^{-1}(\bar{A}_{21} - \bar{B}_{12}\bar{B}_{13}^{-1}\bar{A}_{31}) & 0 & 0 \\ 0 & -\bar{B}_{23}^{-1}\bar{A}_{32} & 0 \end{bmatrix} \bar{x} + \hat{u}_2 \tag{6.55b}$$

then the resulting partially closed-loop system has the form,

$$\dot{\bar{x}} = \begin{bmatrix} \bar{A}_{11} - \bar{B}_{14}\bar{B}_{13}^{-1}\bar{A}_{31} & 0 & \bar{A}_{13} \\ 0 & \bar{A}_{22} - \bar{B}_{24}\bar{B}_{23}^{-1}\bar{A}_{32} & \bar{A}_{23} \\ 0 & 0 & \bar{A}_{33} \end{bmatrix} \bar{x} + \begin{bmatrix} \bar{B}_{11} & \bar{B}_{14} \\ 0 & \bar{B}_{12} \\ 0 & \bar{B}_{13} \end{bmatrix} \hat{u}_1 + \begin{bmatrix} 0 & \bar{B}_{21} \\ \bar{B}_{22} & \bar{B}_{24} \\ 0 & \bar{B}_{23} \end{bmatrix} \hat{u}_2 \tag{6.56}$$

It can be readily seen that the system (6.56), (6.54b) has the desired form of (6.7). Under appropriate assumptions, Proposition 6.1 can be applied to design $\hat{u}_1, \hat{u}_2$ as FPS Structure-Preserving strategies.

It is significant to note that making the dimension of $\bar{B}_{ii}$ as large as possible results in a 'maximally-decoupled' system i.e.; a system in which the decentralized control problems are of the highest possible dimension, and consequently the 'core' problem is of lowest possible dimension [62].

The use of decoupling control introduces a degree of suboptimality if the performance indices are chosen a priori. This is because the decoupling control is chosen from a purely algebraic point of view without any optimality considerations.

We would like to remark that the use of decoupling control requires a degree of mutual cooperation among the DMs. This cannot be guarenteed under the noncooperative Nash concept in general, unless, the resulting advantages constitute a strong enough incentive for the DMs to compensate for the performance loss resulting from the use of decoupling control. But, within a cooperative framework, the use of decoupling control can be readily ensured.

In problems where there is a need for the DMs to use the decoupling control, it will be more appropriate for them to choose their performance indices with respect to the strategies $\hat{u}_i$ after the decoupling has been achieved. Again, this is easier to ensure in a cooperative framework than in a noncooperative framework.

Hence, in situations when the decoupling control has to be used, a semicooperative or cooperative framework is desirable for the application of our techniques.

6.5. Extensions

In this section we shall discuss briefly, extensions of our ideas to many DM problems and cooperative Pareto games.

6.5.1. Many decision maker problems

In situations with more than two DMs there is more than one way to approach the problem; each approach resulting in a different order of simplification. Ideally one would like to identify the individually observable modes, the pair-wise observable modes and so on; and overlap appropriately the input structure of each DM with this observability decomposition. The design of Structure-Preserving Nash strategies would then lead to the solution of low-order control problems, problems where two DMs interact, problems where three DMs interact and so on up to the core problem where all the DMs interact.

In the three DM case the (A,C_1,C_2,C_3) matrices in the observability decomposition form will look like

$$A = \begin{bmatrix} A_{11} & 0 & 0 & A_{14} & 0 & A_{16} & A_{17} \\ 0 & A_{22} & 0 & A_{24} & A_{25} & 0 & A_{27} \\ 0 & 0 & A_{33} & 0 & A_{35} & A_{36} & A_{37} \\ 0 & 0 & 0 & A_{44} & 0 & 0 & A_{47} \\ 0 & 0 & 0 & 0 & A_{55} & 0 & A_{57} \\ 0 & 0 & 0 & 0 & 0 & A_{66} & A_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{77} \end{bmatrix}$$

$$C_1 = [\, C_{11} \quad 0 \quad 0 \quad C_{14} \quad 0 \quad C_{16} \quad C_{17} \,]$$

$$C_2 = [\, 0 \quad C_{22} \quad 0 \quad C_{24} \quad C_{25} \quad 0 \quad C_{27} \,]$$

$$C_3 = [\, 0 \quad 0 \quad C_{33} \quad 0 \quad C_{35} \quad C_{36} \quad C_{37} \,]$$

It can be readily seen that the number of blocks to be identified in the system matrices grows exponentially as the number of DMs increase. Hence for a large number of DMs such a decomposition may be difficult to achieve in practice. The other extreme would be to identify only the modes observable by each DM alone, and consider the rest as commonly observable modes. This will result in only a first order of simplification because the core problem will be of a higher dimension. Of course in practice, depending on the problem, any approach in between these two extremes may be adopted, resulting in different orders of simplification.

6.5.2. Pareto games

Multimodel solutions to cooperative Pareto games based on the structural decompositions of Section 6.2 can be obtained in a straightforward manner. To illustrate this point we shall give below the Structure-Preserving Pareto strategies under the FPS information pattern.

Define the overall system cost as

$$J = \sum_{i=1}^{2} \alpha_i J_i \; ; \quad 0 \le \alpha_i \le 1 \; ; \; \alpha_1 + \alpha_2 = 1 \tag{6.57}$$

Applying the Matrix Minimum Principle, the FPS Structure-Preserving Pareto strategy $\gamma_i^*(\cdot) \in \Gamma_i$, defined by (6.9) for the system (6.7) and performance index (6.57) is obtained as (for i = 1,2),

$$F_{ii}^* = \hat{R}_{ii}^{-1} \hat{B}_{ii}' M_{ii} \tag{6.58a}$$

$$F_{i3}^* = \hat{R}_{ii}^{-1} \hat{B}_{ii}' M_{i3} \tag{6.58b}$$

$$F_{3i}^* = \hat{R}_{ij}^{-1} \hat{B}_{i3}' [\frac{1}{\alpha_i} M_{33} + M_{i3}' L_{i3} L_{33}^{-1}] + \hat{R}_{ij}^{-1} \hat{B}_{i4}' [M_{i3} + M_{ii} L_{i3} L_{33}^{-1}] \tag{6.58c}$$

where

$$M_{ii}\bar{A}_{ii} + \bar{A}_{ii}' M_{ii} + \bar{C}_{ii}'\bar{C}_{ii} - M_{ii}\hat{B}_{ii}\hat{R}_{ii}^{-1}\hat{B}_{ii}' M_{ii} = 0 \tag{6.59a}$$

$$M_{ii}\hat{A}_{i3}^* + M_{i3}\hat{A}_{33}^* + \hat{A}_{ii}^{*\prime} M_{i3} + \bar{C}_{ii}'\bar{C}_{i3} + F_{ii}^{*\prime}\hat{R}_{ii}F_{i3}^* = 0 \tag{6.59b}$$

$$\begin{aligned} M_{33}\hat{A}_{33}^* + \hat{A}_{33}^{*\prime} M_{33} + \sum_{i=1}^{2} \alpha_i (M_{i3}'\hat{A}_{i3}^* + \hat{A}_{i3}^{*\prime} M_{i3} + \bar{C}_{i3}'\bar{C}_{i3} + F_{i3}^{*\prime}\hat{R}_{ii}F_{i3}^*) \\ + \alpha_1 F_{31}^{*\prime}\hat{R}_{12}F_{31}^* + \alpha_2 F_{32}^{*\prime}\hat{R}_{21}F_{32}^* = 0 \end{aligned} \tag{6.59c}$$

$$\hat{A}^*_{ii}L_{i3} + \hat{A}^*_{i3}L_{33} + L_{i3}\hat{A}^{*\prime}_{33} + N_{i3} = 0 \tag{6.60a}$$

$$\hat{A}^*_{33}L_{33} + L_{33}\hat{A}^{*\prime}_{33} + N_{33} = 0 \tag{6.60b}$$

The controllability-observability of the triple $\{(\bar{A}_{ii},\hat{B}_{ii},\bar{C}_{ii});\ i = 1,2\}$ guarantees $\{\mathrm{Re}\ \lambda(\hat{A}^*_{ii}) < 0;\ i = 1,2\}$. For the solution to be well-defined we need only to verify that $\mathrm{Re}\ \lambda(\hat{A}^*_{33}) < 0$. The solution given by (6.58)-(6.60) has features similar to the Nash problem of Section 6.3.1 (like partial noninteraction).

The Structure-Preserving Pareto strategies under the FIS information can be obtained in a similar manner. The solution will have features similar to the Nash problem of Section 6.3.2.

6.6 Applications

Now we shall examine the applicability of our design methodologies to the control of large scale interconnected subsystems and multiarea power systems.

6.6.1. Large scale interconnected subsystems

Consider the large scale system wherein each subsystem is controlled by one DM having his own performance objective. The system considered is of the form

$$\dot{x}_i = A_{ii}x_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} A_{ij}y_j + B_{ii}u_i \tag{6.61a}$$

$$y_i = C_i x_i\ ;\quad i = 1,2,\ .\ ,\ N \tag{6.61b}$$

where the output variables y_i are the interconnection variables. The above problem has been considered in [55] as a single DM problem. We shall demonstrate that when viewed as a multiple DM problem, the techniques developed in this paper can be applied for optimal strategy design.

For simplicity we shall consider the two subsystem case ($N=2$). As in [55] suppose that each subsystem is transformed with respect to its own output. The transformed system can be represented as

$$\begin{bmatrix} \dot{y}_1 \\ \dot{x}_{1r} \\ \hline \dot{y}_2 \\ \dot{x}_{2r} \end{bmatrix} = \left[\begin{array}{cc|cc} F_{11}^{(1)} & F_{12}^{(1)} & F^{12} & 0 \\ F_{21}^{(1)} & F_{22}^{(2)} & F_r^{12} & 0 \\ \hline F^{21} & 0 & F_{11}^{(2)} & F_{12}^{(2)} \\ F_r^{21} & 0 & F_{21}^{(2)} & F_{22}^{(2)} \end{array}\right] \begin{bmatrix} y_1 \\ x_{1r} \\ \hline y_2 \\ x_{2r} \end{bmatrix} + \begin{bmatrix} G_{11} \\ G_{12} \\ \hline 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ \hline G_{22} \\ G_{21} \end{bmatrix} u_2 \quad (6.62)$$

By a simple reordering of variables (6.62) can be written as,

$$\begin{bmatrix} \dot{x}_{1r} \\ \hline \dot{x}_{2r} \\ \hline \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \left[\begin{array}{c|c|cc} F_{22}^{(1)} & 0 & F_{21}^{(1)} & F_r^{12} \\ \hline 0 & F_{22}^{(2)} & F_r^{21} & F_{21}^{(2)} \\ \hline F_{12}^{(1)} & 0 & F_{11}^{(1)} & F^{12} \\ 0 & F_{12}^{(2)} & F^{21} & F_{11}^{(2)} \end{array}\right] \begin{bmatrix} x_{1r} \\ \hline x_{2r} \\ \hline y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} G_{12} \\ \hline 0 \\ \hline G_{11} \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \hline G_{21} \\ \hline 0 \\ G_{22} \end{bmatrix} u_2 \quad (6.63)$$

Now, by making an appropriate input space transformation [55] and letting DMi use his own residual state feedback to cancel the terms $F_{12}^{(i)} x_{ir}$, we obtain a system which is in the familiar observability decomposition form. The interconnection

variables y_i represent the variables observable by both the DMs, and the residual variables x_{ir} represent the variables observable by DMi alone.

Suppose DMi chooses his performance index as,

$$J_i = \frac{1}{2}\int_0^\infty (y_i'y_i + x_{ir}'Q_ix_{ir} + \hat{u}_i'R_i\hat{u}_i)dt \ ; \qquad i = 1,2 \tag{6.64}$$

where, u_i = decoupling control + $\hat{u}_i$ then, assuming that each DM has access to all the interconnection variables and his own subsystem variables, Structure-Preserving linear Feedback Nash strategies $\hat{u}_i$ can be generated from multimodel solutions of Proposition 6.1.

6.6.2. Two-area Power System

This example has been considered in [55] in the single DM context. Here we shall assume that each area is under a different control authority. We shall first transform the system into our desired form given by (6.7), and then obtain Pareto strategies on solving equations (6.58)-(6.60).

A two-area power system with each area containing two thermal plants is constructed from [60]. The system is modeled by

$$\begin{aligned} \dot{x} &= Ax + B_1u_1 + B_2u_2 \\ y_i &= C_ix \ ; \quad i = 1,2 \end{aligned} \tag{6.65}$$

where $x \in R^{19}$, $u_1 \in R^2$, $u_2 \in R^2$, $y_1 \in R^2$, $y_2 \in R^2$. The state, control and output variables are defined in Appendix D.

The system matrices are given by,

$$A = \begin{bmatrix} A_{11}^{(1)} & 0 & A_{13}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & A_{22}^{(1)} & A_{23}^{(1)} & 0 & 0 & 0 & 0 \\ A_{31}^{(1)} & A_{32}^{(1)} & -0.1124 & -0.083 & 0 & 0 & 0 \\ 0 & 0 & 22.21 & 0 & -22.21 & 0 & 0 \\ 0 & 0 & 0 & 0.083 & -0.1124 & A_{31}^{(2)} & A_{32}^{(2)} \\ 0 & 0 & 0 & 0 & A_{13}^{(2)} & A_{11}^{(2)} & 0 \\ 0 & 0 & 0 & 0 & A_{23}^{(2)} & 0 & A_{22}^{(2)} \end{bmatrix}$$

$$A_{11}^{(1)} = A_{11}^{(2)} = A_{22}^{(1)} = A_{22}^{(2)} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 4.75 & -5 & 0 & 0 \\ 0 & 0.167 & -0.167 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$A_{13}^{(1)} = A_{13}^{(2)} = A_{23}^{(1)} = A_{23}^{(2)} = [-4 \quad 0 \quad 0 \quad 0]'$$

$$A_{31}^{(1)} = A_{31}^{(2)} = A_{32}^{(1)} = A_{32}^{(2)} = [0 \quad 0.01 \quad 0.0093 \quad 0.014]$$

$$B_1 = \left[\begin{array}{cc} B_{11}^{(1)} & 0 \\ 0 & B_{22}^{(1)} \\ \hline \multicolumn{2}{c}{0_{11x2}} \end{array}\right] \; ; \; B_2 = \left[\begin{array}{cc} \multicolumn{2}{c}{0_{11x2}} \\ \hline B_{11}^{(2)} & 0 \\ 0 & B_{22}^{(2)} \end{array}\right] \; ;$$

$$B_{11}^{(1)} = B_{11}^{(2)} = B_{22}^{(1)} = B_{22}^{(2)} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \left[\begin{array}{c|cc|c} 0_{2x8} & \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} & & 0_{2x9} \end{array}\right] \; ; \; C_2 = \left[\begin{array}{c|cc|c} 0_{2x9} & \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} & & 0_{2x8} \end{array}\right] .$$

After two steps of chained aggregation [54,55] and one input space transformation, we obtain the following representation:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} F_{11}^{(1)} & 0 & F_{13} \\ 0 & F_{11}^{(2)} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} G_{11}^{(1)} & G_{12}^{(1)} \\ 0 & 0 \\ 0 & G_{31} \end{bmatrix} \begin{bmatrix} \bar{u}_1^{(1)} \\ \bar{u}_2^{(1)} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ G_{11}^{(2)} & G_{12}^{(2)} \\ 0 & G_{32} \end{bmatrix} \begin{bmatrix} \bar{u}_1^{(2)} \\ \bar{u}_2^{(2)} \end{bmatrix}$$

$$\bar{x}_1 \in R^6 \; ; \quad \bar{x}_2 \in R^6 \; ; \quad \bar{x}_3 \in R^7 \tag{6.66}$$

where

$$F_{11}^{(i)} = \begin{bmatrix} -5 & 4.75 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -0.167 & 0.167 & 0 \\ 0 & 0 & 0 & 0 & -5 & 4.75 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \; ; \; i=1,2$$

$$F_{33} = \begin{bmatrix} -0.1124 & -0.083 & 0 & 1 & 0 & 0 & 0 \\ 22.21 & 0 & -22.21 & 0 & 0 & 0 & 0 \\ 0 & 0.083 & -0.1124 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ -0.38 & 0 & 0 & 0 & -0.167 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & -0.38 & 0 & 0 & 0 & -0.167 \end{bmatrix}$$

$$F_{31} = \begin{bmatrix} 0_{4x6} \\ \text{------} \\ F_2^{(1)} \\ \text{------} \\ 0_{2x6} \end{bmatrix} \quad ; \qquad F_{32} = \begin{bmatrix} 0_{6x6} \\ \text{------} \\ F_2^{(2)} \end{bmatrix}$$

$$F_2^{(i)} = [0.136 \quad -0.222 \quad 0 \quad 0 \quad 0.136 \quad -0.222] \qquad ; \qquad i=1,2$$

$$F_{13} = \left[\; 0_{6x3} \;\vdots\; D \;\vdots\; 0_{6x3} \; \right] : \; F_{23} = \left[\; 0_{6x5} \;\vdots\; D \;\vdots\; 0_{6x1} \; \right]$$

$$D = [\,0 \quad -4 \quad 0 \quad 0 \quad 0 \quad -4\,]'$$

$$G_{11}^{(i)} = [\,0 \quad -4 \quad 0 \quad 0 \quad 0 \quad 4\,]' \qquad ; \qquad i=1,2$$

$$G_{12}^{(i)} = [\,0 \quad 4 \quad 0 \quad 0 \quad 0 \quad 0\,]' \qquad ; \qquad i=1,2$$

$$G_{31} = [\,0 \quad 0 \quad 0 \quad 0 \quad 0.19 \quad 0 \quad 0\,]'$$

$$G_{32} = [\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.19\,]'$$

Now we need to apply a decoupling control to cancel out the terms F_{31} and F_{32} in (6.66). The decoupling control is chosen to be,

$$\bar{u}_2^{(i)} = [-0.716 \quad 1.168 \quad 0 \quad 0 \quad -0.716 \quad 1.168]\, \bar{x}_i + \hat{u}_2^{(i)} \quad ; \quad i=1,2. \tag{6.67}$$

Substituting (6.67) in (6.66) we obtain,

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} \bar{F}_{11}^{(1)} & 0 & F_{13} \\ 0 & \bar{F}_{11}^{(2)} & F_{23} \\ 0 & 0 & F_{33} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} G_{11}^{(1)} & G_{12}^{(1)} \\ 0 & 0 \\ 0 & G_{31} \end{bmatrix} \begin{bmatrix} \bar{u}_1^{(1)} \\ \hat{u}_2^{(1)} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ G_{11}^{(2)} & G_{12}^{(2)} \\ 0 & G_{32} \end{bmatrix} \begin{bmatrix} \bar{u}_1^{(2)} \\ \hat{u}_2^{(2)} \end{bmatrix} \tag{6.68}$$

where

$$\bar{F}_{11}^{(1)} = \bar{F}_{11}^{(2)} = \begin{bmatrix} -5 & 4.75 & 0 & 0 & 0 & 0 \\ -2.864 & 2.612 & 0 & 0 & -2.864 & 4.612 \\ 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -0.167 & 0.167 & 0 \\ 0 & 0 & 0 & 0 & -5 & 4.75 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Now the system is precisely in a form suitable for our design techniques. The frequency deviations in the two areas and the tie-line power flow comprise part of the core variables $\bar{x}_3$. The variables $\bar{x}_1, \bar{x}_2$ are the residual variables associated with each area.

The nineteenth-order game in its original form (6.65) may be computationally intractable. But in the form (6.68), and allowing only FPS Structure-Preserving strategies, we need only to solve two sixth-order optimal control problems, and one seventh-order problem where the two DMs interact.

For the Pareto-optimal design, the cost functionals are chosen to be

$$J_i = \frac{1}{2}\int_0^\infty (\bar{x}_i' Q_{ii}\bar{x}_i + \bar{x}_3' Q_{i3}\bar{x}_3 + \hat{u}_i'\hat{R}_i\hat{u}_i)\,dt \qquad ; \qquad i=1,2 \;;$$

with

$$Q_{11} = \text{diag}\,(10,\ 10,\ 10,\ 10,\ 1,\ 1)$$

$$Q_{22} = \text{diag}\,(12,\ 15,\ 10,\ 5,\ 5,\ 5)$$

$$Q_{13} = \text{diag}\,(10,\ 7,\ 0,\ 0,\ 0,\ 0,\ 0)$$

$$Q_{23} = \text{diag}\,(0,\ 5,\ 10,\ 0,\ 0,\ 0,\ 0,\ 0)$$

$$\hat{R}_1 = \text{diag}\,(10,\ 25)$$

$$\hat{R}_2 = \text{diag}\,(5,\ 20) \qquad ; \qquad \text{Cov}\,(\bar{x}_0) = N = I.$$

<u>Case 1</u>: Pareto cost $J = \frac{2}{5}J_1 + \frac{3}{5}J_2$

The optimal gains F_{ii}^* are first obtained from optimal control problems (6.58a), (6.59a)

$$F_{11}^* = [-0.167 \quad -0.722 \quad 0.0132 \quad 0.571 \quad 0.043 \quad 0.13]$$

$$F_{22}^* = [-0.402 \quad -1.082 \quad 0.017 \quad 0.528 \quad 0.042 \quad 0.714].$$

Then the optimal gains F_{i3}^*, F_{3i}^* are obtained from the coupled equations (6.58b,c), (6.59b,c) and (6.60),

$$F_{13}^* = [-4.65 \quad -16.28 \quad 1.37 \quad 12.44 \quad -5.53 \quad 0 \quad 0]$$

$$F_{23}^* = [-6.62 \quad -21.35 \quad 3.04 \quad 0 \quad 0 \quad -10.92 \quad 4.15]$$

$$F_{31}^* = [16.5 \quad 0.589 \quad -16.44 \quad -0.653 \quad -0.068 \quad 0 \quad 0]$$

$$F_{32}^* = [-17.08 \quad 0.568 \quad 16.33 \quad 0 \quad 0 \quad -8.21 \quad -0.121].$$

The closed-loop eigenvalues turn out to be $-0.2 \pm j0.51$, $-0.24 \pm j0.48$, $-0.39 \pm j0.05$, $-0.52 \pm j0.07$, $-1.03 \pm j1.5$, -1.99, -2, -2.09, -2.21, -2.21, $-5.18 \pm j1.92$, $-7.09 \pm j1.96$.

The feedback strategies are obtained as,

$$u_1^* = \left[\begin{array}{ccccccccccc|c} -0.722 & 0.167 & 0.263 & -0.591 & -0.13 & -0.043 & -0.571 & -0.013 & -0.221 & 0.773 & -0.065 & \\ -1.168 & 0.716 & 0.03 & -0.03 & -1.168 & -0.716 & 0 & 0 & -0.784 & -0.028 & 0.781 & O_{2x8} \end{array}\right] x$$

$$u_2^* = \left[\begin{array}{c|ccccccccccc} & 0.314 & -1.014 & -0.144 & -1.082 & 0.402 & -0.197 & 0.519 & -0.714 & -0.042 & -0.528 & -0.017 \\ O_{2x8} & 0.811 & -0.027 & 0.776 & -1.168 & 0.716 & -0.057 & 0.037 & -1.168 & -0.716 & 0 & 0 \end{array}\right] x$$

<u>Case 2</u>: Pareto cost $J = 0.1\, J_1 + 0.9\, J_2$

The gains F^*_{ii} do not change and remain the same as before. The gains F^*_{i3}, F^*_{3i} are obtained as

$$F^*_{13} = [-6.72 \quad -18.33 \quad 4.47 \quad 10.92 \quad -6.71 \quad 0 \quad 0]$$

$$F^*_{23} = [-5.91 \quad -19.38 \quad 3.72 \quad 0 \quad 0 \quad -13.08 \quad 3.17]$$

$$F^*_{31} = [21.26 \quad 4.715 \quad -20.38 \quad -5.05 \quad -2.313 \quad 0 \quad 0]$$

$$F^*_{32} = [-15.15 \quad 0.481 \quad 14.77 \quad 0 \quad 0 \quad -0.514 \quad -0.131] \; .$$

The closed-loop eigenvalues turn out to be $-0.13\pm j0.56$, -0.172, $-0.39\pm j0.05$, $-0.52\pm j0.07$, -0.61, $-1.03\pm j1.15$, -1.99, -2.0, -2.21, -2.21, -3.12, $-5.18\pm j1.92$, $-7.09\pm j1.96$.

The feedback strategies are obtained as,

$$u^*_1 = \left[\begin{array}{ccccccccccc|c} -0.722 & 0.167 & 0.412 & -0.716 & -0.13 & -0.043 & -0.571 & -0.013 & -0.395 & 1.012 & -0.111 & \\ -1.168 & 0.716 & 0.068 & -0.11 & -1.168 & -0.716 & 0 & 0 & -0.913 & -0.066 & 0.884 & 0_{2\times 8} \end{array}\right] x$$

$$u^*_2 = \left[\begin{array}{c|ccccccccccc} & 0.198 & 0.812 & -0.133 & -1.082 & 0.402 & -0.156 & 0.363 & -0.714 & -0.042 & -0.528 & -0.017 \\ 0_{2\times 8} & 0.626 & -0.012 & 0.494 & -1.168 & 0.716 & -0.032 & 0.024 & -1.168 & -0.716 & 0 & 0 \end{array}\right] x$$

<u>Case 3</u>: Pareto cost $J = 0.9\, J_1 + 0.1\, J_2$

F_{ii}^* remains the same. F_{i3}^*, F_{3i}^* are obtained as,

$$F_{13}^* = [-2.82 \quad -13.19 \quad 1.11 \quad 13.73 \quad -3.88 \quad 0 \quad 0]$$

$$F_{23}^* = [-7.75 \quad -25.62 \quad 6.51 \quad 0 \quad 0 \quad -11.21 \quad 5.34]$$

$$F_{31}^* = [13.91 \quad 0.366 \quad -14.48 \quad -0.489 \quad -0.041 \quad 0 \quad 0]$$

$$F_{32}^* = [-18.19 \quad 0.614 \quad 17.56 \quad 0 \quad 0 \quad -1.112 \quad -0.291].$$

The closed-loop eigenva'ues turn out to be $-0.1\pm j0.66$, -0.158, $-0.39\pm j0.05$, $-0.52\pm j0.07$, -0.542, $-0.98\pm j1.52$, -1.99, -2.16, -2.21, -2.21, $-5.18\pm j1.92$, $-7.09\pm j1.96$.

The feedback strategies are obtained as,

$$u_1^* = \left[\begin{array}{cccccccc} -0.722 & 0.167 & 0.115 & -0.482 & -0.13 & -0.043 & -0.571 & -0.013 \\ -1.168 & 0.716 & 0.019 & -0.023 & -1.168 & -0.716 & 0 & 0 \end{array}\right.$$

$$\left.\begin{array}{ccc|} -0.106 & 0.518 & -0.06 \\ -0.613 & -0.014 & 0.593 \end{array}\; O_{2\times8}\right] x$$

$$u_2^* = \left[\; O_{2\times8} \;\begin{array}{|ccccccc} 0.523 & 1.131 & -0.393 & -1.082 & 0.402 & -0.403 & 0.626 \\ 0.928 & -0.047 & 1.022 & -1.168 & 0.716 & -0.109 & 0.042 \end{array}\right.$$

$$\left.\begin{array}{cccc} -0.714 & -0.042 & -0.528 & -0.017 \\ -1.168 & -0.716 & 0 & 0 \end{array}\right] x$$

Notice that the strategy of each DM requires a knowledge of the states of his own area and only the frequency deviation of the other area; a feature desirable from implementation point-of-view. The time responses of the tie-line power flow and frequency deviations of the two areas are plotted in Figures 6.1-6.3. It can be seen that the response of the frequency deviation corresponding to the area weighted lightly in the Pareto cost is more oscillatory, which is what one would expect. The response of the tie-line power flow does not change significantly in the three cases.

6.7. Conclusions

In this chapter we have examined the role of the observability structure in multiple decision maker problems. By identifying explicitly the observability decomposition induced by the observation sets of the DMs, and by overlapping appropriately the input structure of each DM, we have shown that the design of Structure-Preserving Feedback Nash strategies leads to multimodel solutions. Under the FPS information pattern, the multimodel solutions are shown to admit partial noninteraction among the DMs. Under the FIS information pattern, Structure-Preserving strategies involving only linear static output feedback do not exist in general. When the output matrices split so that there are separate observation channels for the individually and commonly observable modes, Structure-Preserving strategies do exist and are again generated from multimodel solutions. But in this case, the solution is completely interacting unless certain conditions on the statistics of the state variables are satisfied. When the output matrices do not split, the FPS Structure-

Preserving strategies can be synthesized using observers with arbitrarily fast dynamics. This strategy has the property of being near-equilibrium and asymptotic Nash. When the system is completely observable by each DM, the observability decomposition can be induced by using the decoupling controls. But in such situations, a semi-cooperative or cooperative framework is desirable.

Applications to the control of large scale interconnected subsystems and control of multiarea power systems have been examined; and extensions to many DM problems and cooperative Pareto games have been discussed.

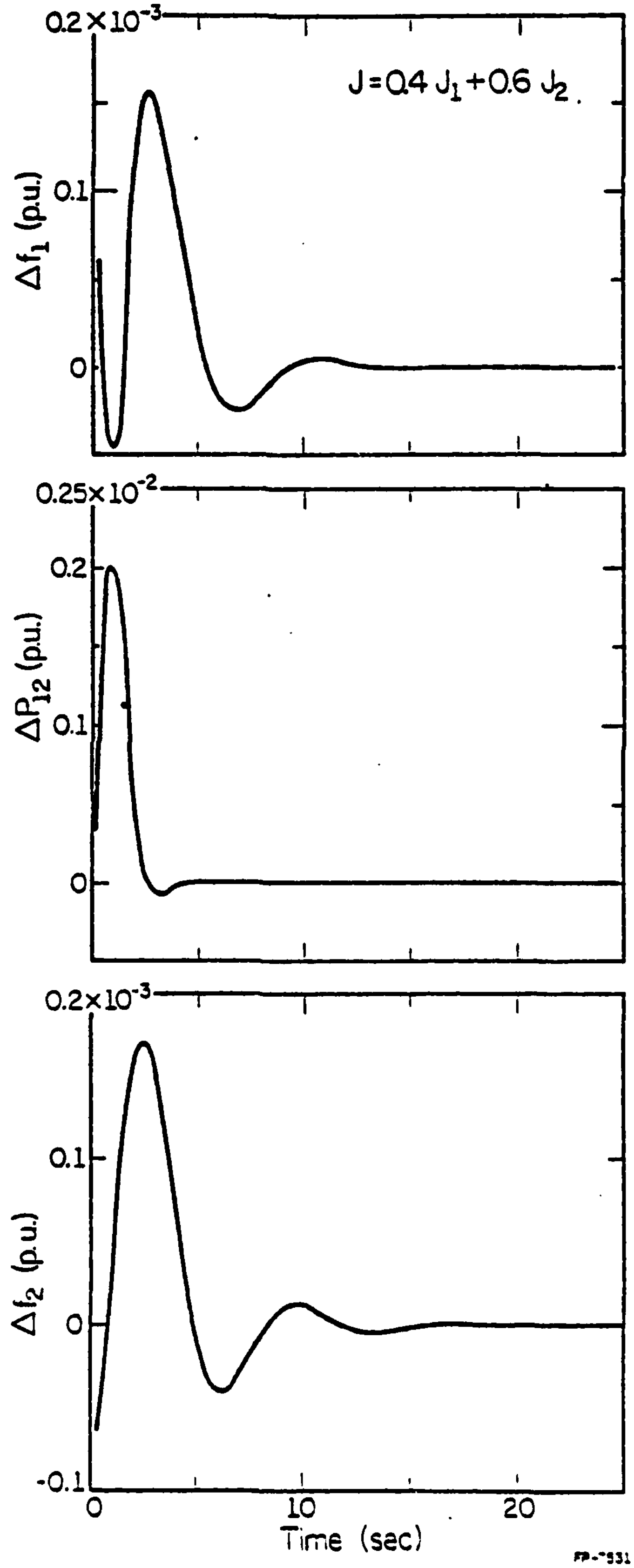

Fig. 6.1. Time responses for the case $J = 0.4\ J_1 + 0.6\ J_2$.

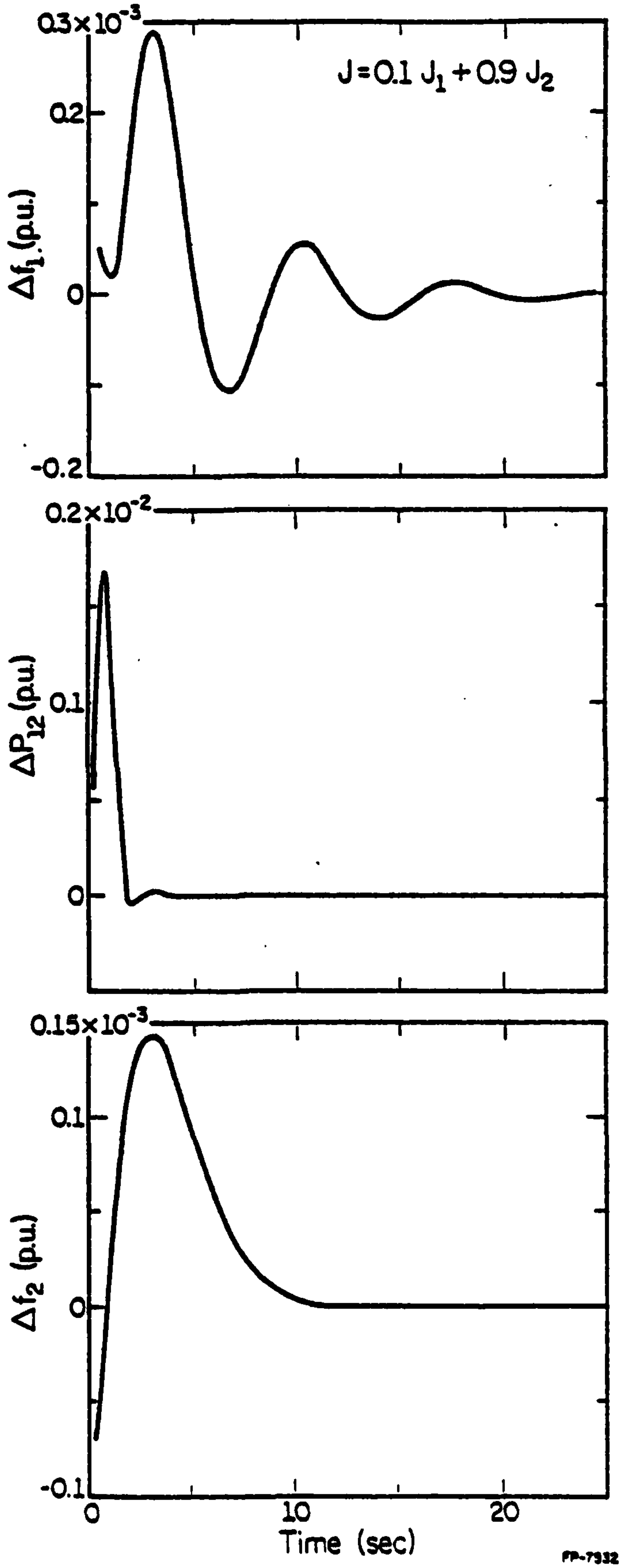

Fig. 6.2. Time responses for the case $J = 0.1\ J_1 + 0.9\ J_2$.

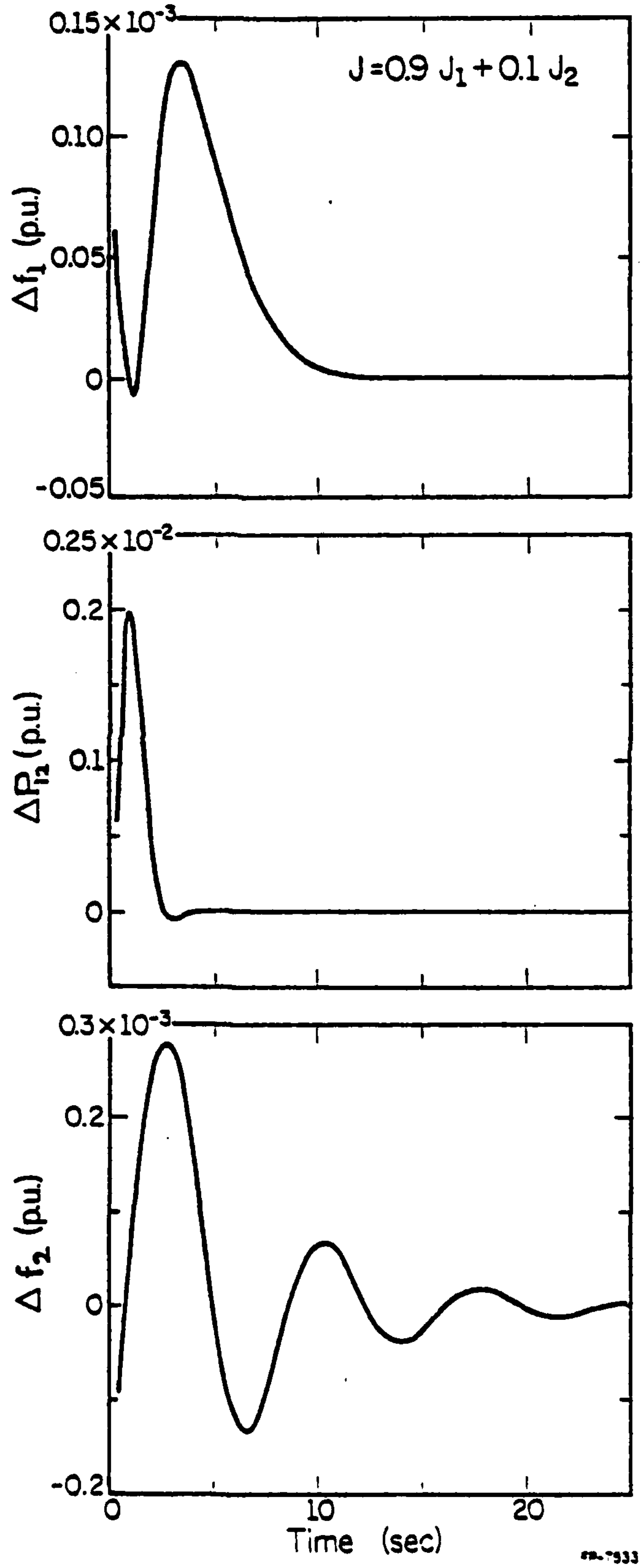

Fig. 6.3. Time responses for the case J = 0.9 J_1 + 0.1 J_2.

CHAPTER 7

CONCLUSIONS

The main thrust of this thesis has been towards analyzing the interaction between model simplification and control strategy design in a multimodel context. We have studied several realistic situations which allow the decision makers to use different simplified models of the system.

In Chapters 2-4, we have established the well-posedness of multimodel generation by 'k-th parameter perturbation' for classes of linear multiparameter singularly perturbed systems. In Chapter 2 we have considered deterministic models without the weak-coupling assumption on the fast subsystems and obtained near-optimal decentralized strategies from multiple noncausal reduced-order models. In Chapters 3 and 4 we have considered stochastic version of the model considered in [15,16] with decentralized observations for the decision makers. In Chapter 3 we developed multimodel solutions for a Nash game with prespecified finite-dimensional compensator structure for each decision maker. In Chapter 4 we developed multimodel solutions for team problems with sampled observations for the decision makers. Both the static team problem and the dynamic team problem with one-step-delay observation-sharing pattern have been considered.

In Chapter 5 we have considered the average-cost-per-stage problem for finite-state Markov chains. The focus was on obtaining near-optimal incentive policies for controlled Markov models consisting of N weakly-coupled groups of strongly-interacting states. A hierarchical

algorithm, which allowed for multimodeling on the part of the 'local' decision makers, has been proposed for computing the near-optimal incentive policies.

In Chapter 6 we have taken an aggregation-based approach to multimodeling. Based on input-output considerations, we restructured the problem in such a way that the optimal solution within a class of admissible strategies (defined as Structure-Preserving strategies) could be obtained from multiple reduced-order models. In some cases, the solution has the desirable feature of partial noninteraction among the decision makers.

The main contribution of this thesis has been towards strengthening and extending the multimodeling concept beyond the framework within which it was originally introduced in [14,15]. We have achieved this by examining three different approaches to multimodeling. The first approach (same as in [14,15]) has been to establish the validity of a rational multimodel generation scheme which is chosen a-priori. The results of Chapters 2-4 have strengthened this approach by establishing the 'robustness' of multimodel generation by 'k-th parameter perturbation ' proposed in [15], to a class of solution concepts and information patterns. The next two approaches have extended the multimodeling concept beyond the framework of [14,15]. The second approach, taken in Chapter 5, has been to develop a numerical algorithm for computing near-optimal policies, which allows the decision makers to use multiple reduced-order models. The final approach, taken in Chapter 6, has been to induce multimodel solutions by

an appropriate restructuring of the problem and a suitable choice of admissible strategies. The results of this thesis have revealed the interplay between model simplification tools like time-scales, weak-coupling, controllability-observability, and strategy design concepts like team, Nash and Stackelberg.

There are many possible directions for further research along the lines of the results obtained in this thesis. For the models considered in Chapters 2-4, Stackelberg problems with dynamic information (with/without memory) for the leader [64] can be analyzed. Also multimodeling possibilities can be explored for nonlinear deterministic and stochastic models of the type considered in [67,68]. For Markovian models considered in Chapter 5, it will be rather straightforward to analyze the finite horizon and infinite horizon discounted cost problems with state information. A nontrivial extension would be to problems with decentralized imperfect information for the decision makers [66]. In the aggregation-based approach of Chapter 6, we have assumed an 'exact' system decomposition. A possibly more practical problem would be to consider situations when there is only a 'weak' decomposition of the system. A perturbational decomposition-aggregation approach could be developed to obtain near-optimal policies for such problems.

Possibilities for a multimodel design approach based on overlapping decompositions [69] and state vector partitioning [70] can also be explored.

APPENDIX A

MATRIX DEFINITIONS APPEARING IN CHAPTER 2

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10}/\varepsilon_1 & A_{11}/\varepsilon_1 & A_{12}/\varepsilon_1 \\ A_{20}/\varepsilon_2 & A_{21}/\varepsilon_2 & A_{22}/\varepsilon_2 \end{bmatrix} ; \quad B_i = \begin{bmatrix} B_{oi} \\ B_{1i}/\varepsilon_1 \\ B_{2i}/\varepsilon_2 \end{bmatrix} ; \quad Q_i = \begin{bmatrix} Q_{oi} & 0 & 0 \\ 0 & Q_{ii}\delta_{i1} & 0 \\ 0 & 0 & Q_{ii}\delta_{i2} \end{bmatrix}$$

where $\delta_{ij} = 0; \quad i \neq j$

$= 1; \quad i = j$

$$S_i = B_i R_{ii}^{-1} B_i' \; ; \qquad S_{ij} = B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j'$$

$$A_{oo}^{(i)} = A_{oo} - A_{oj} A_{jj}^{-1} A_{jo} \; ; \qquad A_{oi}^{(i)} = A_{oi} - A_{oj} A_{jj}^{-1} A_{ji}$$

$$B_{oi}^{(i)} = B_{oi} - A_{oj} A_{jj}^{-1} B_{ji} \; ; \qquad B_{oj}^{(i)} = B_{oj} - A_{oj} A_{jj}^{-1} B_{jj}$$

$$A_{io}^{(i)} = A_{io} - A_{ij} A_{jj}^{-1} A_{jo} \; ; \qquad A_{ii}^{(i)} = A_{ii} - A_{ij} A_{jj}^{-1} A_{ji}$$

$$B_{ii}^{(i)} = B_{ii} - A_{ij} A_{jj}^{-1} B_{ji} \; ; \qquad B_{ij}^{(i)} = B_{ij} - A_{ij} A_{jj}^{-1} B_{jj}$$

$$Q_{oj}^{(i)} = Q_{oj} + (A_{jj}^{-1} A_{jo})' Q_{jj} (A_{jj}^{-1} A_{jo}) \; ; \qquad Q_{jj}^{(i)} = (A_{jj}^{-1} A_{ji})' Q_{jj} (A_{jj}^{-1} A_{ji})$$

$$Q_{ij}^{(i)} = (A_{jj}^{-1} A_{jo})' Q_{jj} (A_{jj}^{-1} A_{ji}) \; ; \qquad R_{jj}^{(i)} = R_{jj} + (A_{jj}^{-1} B_{jj})' Q_{jj} (A_{jj}^{-1} B_{jj})$$

$$R_{ji}^{(i)} = R_{ji} + (A_{jj}^{-1} B_{ji})' Q_{jj} (A_{jj}^{-1} B_{ji}) \; ; \qquad S_j^{(i)} = (A_{jj}^{-1} A_{jo})' Q_{jj} (A_{jj}^{-1} B_{ji})$$

$$P_j^{(i)} = (A_{jj}^{-1} A_{jo})' Q_{jj} (A_{jj}^{-1} B_{jj}) \; ; \qquad T_i^{(i)} = (A_{jj}^{-1} A_{ji})' Q_{jj} (A_{jj}^{-1} B_{ji})$$

$$T_j^{(i)} = (A_{jj}^{-1} A_{ji})' Q_{jj} (A_{jj}^{-1} B_{jj}) \; ; \qquad P_{jj}^{(i)} = (A_{jj}^{-1} B_{jj})' Q_{jj} (A_{jj}^{-1} B_{ji})$$

$$\hat{A}_{oi}^{(i)} = A_{oi}^{(i)} - B_{oi}^{(i)} M_{if}^{(i)} - B_{oj}^{(i)} M_{jf}^{(i)} \; ; \qquad \hat{A}_{ii}^{(i)} = A_{ii}^{(i)} - B_{ii}^{(i)} M_{if}^{(i)} - B_{ij}^{(i)} M_{jf}^{(i)}$$

$$\hat{Q}_{ii} = Q_{ii} + M_{if}^{(i)\prime} R_{ii} M_{if}^{(i)} + M_{jf}^{(i)\prime} R_{ij} M_{jf}^{(i)} \; ; \qquad \hat{Q}_{ij}^{(i)} = Q_{ij}^{(i)} - S_j^{(i)} M_{if}^{(i)} - P_j^{(i)} M_{jf}^{(i)}$$

$$\hat{Q}^{(i)}_{jj} = Q^{(i)}_{jj} - T^{(i)}_{i} M^{(i)}_{if} - M^{(i)}_{if} T^{(i)'}_{i} + M^{(i)'}_{jf} R^{(i)}_{jj} M^{(i)}_{jf} + M^{(i)'}_{if} R^{(i)}_{ji} M^{(i)}_{if}$$

$$+ M^{(i)'}_{jf} P^{(i)}_{jj} M^{(i)}_{if} + M^{(i)'}_{if} P^{(i)'}_{jj} M^{(i)}_{jf}$$

$$\hat{T}^{(i)}_{i} = T^{(i)}_{i} - M^{(i)'}_{if} R^{(i)}_{ji} - M^{(i)}_{jf} P^{(i)}_{jj} \quad ; \quad \hat{T}^{(i)}_{j} = T^{(i)}_{j} - M^{(i)'}_{jf} R^{(i)}_{jj} - M^{(i)'}_{if} P^{(i)'}_{jj}$$

$$A^{(i)}_{os} = A^{(i)}_{oo} - \hat{A}^{(i)}_{oi} \hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io} \quad ; \quad B^{(i)}_{is} = B^{(i)}_{oi} - \hat{A}^{(i)^{-1}}_{oi} \hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ii}$$

$$B^{(i)}_{js} = B^{(i)}_{oj} - \hat{A}^{(i)}_{oi} \hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ij}$$

$$Q^{(i)}_{is} = Q_{oi} + (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io})' \hat{Q}_{ii} (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io})$$

$$S^{(i)}_{is} = (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io})' M^{(i)'}_{if} R_{ii} + (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io})' \hat{Q}_{ii} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ii})$$

$$P^{(i)}_{is} = (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io})' M^{(i)'}_{jf} R_{ij} + (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io})' \hat{Q}_{ii} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ij})$$

$$R_{ijs} = R_{ij} + (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ij})' M^{(i)'}_{jf} R_{ij} + R_{ij} M^{(i)}_{jf} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ij})$$

$$+ (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ij})' \hat{Q}_{ii} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ij})$$

$$P^{(i)}_{iis} = (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ii})' \hat{Q}_{ii} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ij}) + (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ii})' M^{(i)'}_{jf} R_{ij} + R_{ii} M^{(i)}_{if} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ij})$$

$$Q^{(i)}_{js} = Q^{(i)}_{oj} - \hat{Q}^{(i)}_{ij} (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io}) - (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io})' \hat{Q}^{(i)'}_{ij} + (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io})' \hat{Q}^{(i)}_{jj} (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io})$$

$$S^{(i)}_{js} = S^{(i)}_{j} - \hat{Q}^{(i)}_{ij} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ii}) - (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io})' \hat{T}^{(i)}_{i} + (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io})' \hat{Q}^{(i)}_{jj} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ii})$$

$$P^{(i)}_{js} = P^{(i)}_{j} - \hat{Q}^{(i)}_{ij} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ij}) - (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io})' \hat{T}^{(i)}_{j} + (\hat{A}^{(i)^{-1}}_{ii} A^{(i)}_{io})' \hat{Q}^{(i)}_{jj} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ij})$$

$$R^{(i)}_{jis} = R^{(i)}_{ji} - (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ii})' \hat{T}^{(i)}_{i} - \hat{T}^{(i)'}_{i} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ii}) + (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ii})' \hat{Q}^{(i)}_{jj} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ii})$$

$$P^{(i)}_{jjs} = P^{(i)}_{jj} - (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ij})' \hat{T}^{(i)}_{i} - \hat{T}^{(i)'}_{j} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ii}) + (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ij})' \hat{Q}^{(i)}_{jj} (\hat{A}^{(i)^{-1}}_{ii} B^{(i)}_{ii})$$

APPENDIX B

MATRIX DEFINITIONS APPEARING IN CHAPTER 3

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10}/\varepsilon_1 & A_{11}/\varepsilon_1 & \varepsilon_{11}A_{12}/\varepsilon_1 \\ A_{20}/\varepsilon_2 & \varepsilon_{22}A_{21}/\varepsilon_2 & A_{22}/\varepsilon_2 \end{bmatrix} ; \quad B_1 = \begin{bmatrix} B_{01} \\ B_{11}/\varepsilon_1 \\ 0 \end{bmatrix}; \quad B_2 = \begin{bmatrix} B_{02} \\ 0 \\ B_{22}/\varepsilon_2 \end{bmatrix} ;$$

$$L = \begin{bmatrix} L_o \\ L_1/\sqrt{\varepsilon_1} \\ L_2/\sqrt{\varepsilon_2} \end{bmatrix}$$

$$C_1 = [\bar{C}_{01} \quad \frac{1}{\sqrt{\varepsilon_1}}\bar{C}_1 \quad 0] = \begin{bmatrix} C_{01} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\varepsilon_1}} C_{11} & 0 \end{bmatrix}$$

$$C_2 = [\bar{C}_{02} \quad 0 \quad \frac{1}{\sqrt{\varepsilon_2}}\bar{C}_2] = \begin{bmatrix} C_{02} & 0 & \\ 0 & 0 & \frac{1}{\sqrt{\varepsilon_1}} C_{22} \end{bmatrix}$$

$$S_i = B_i R_i^{-1} B_i' \ ; \quad S_{oi} = B_{oi} R_i^{-1} B_{oi}' \ ; \quad \tilde{S}_{oi} = B_{oi} R_i^{-1} B_{ii}' \ ; \quad S_{ii} = B_{ii} R_i^{-1} B_{ii}'$$

$$\Gamma_1 = \text{block diag } [\bar{\Gamma}_{01}, \varepsilon_1\bar{\Gamma}_1, 0] \ ; \qquad \Gamma_2 = \text{block diag}[\bar{\Gamma}_{02}, 0, \varepsilon_2\bar{\Gamma}_2]$$

$$Q_1 = \text{block diag}[\bar{Q}_{01}, \bar{Q}_1, 0] \quad ; \qquad Q_2 = \text{block diag}[\bar{Q}_{02}, 0, \bar{Q}_2]$$

$$V_i = \text{block diag}[V_{oi}, V_{ii}] \quad ; \qquad \circledast = \text{block diag}[W, V_1, V_2]$$

$$V_{ij} = \text{block diag}[V_{oi}, V_{ii}] \quad ; \qquad T_{ii} = C_{ii}' V_{ii}^{-1} C_{ii}$$

$$F = \begin{bmatrix} A-S_1K_1-S_2K_2 & S_1K_1 & S_2K_2 \\ A-F_1-S_2K_2 & F_1-G_1C_1 & S_2K_2 \\ A-F_2-S_1K_1 & S_1K_1 & F_2-G_2C_2 \end{bmatrix} ; \quad B = \begin{bmatrix} -L & 0 & 0 \\ -L & G_1 & 0 \\ -L & 0 & G_2 \end{bmatrix}$$

$$A_s = A_{00}-A_{01}A_{11}^{-1}A_{10}-A_{02}A_{22}^{-1}A_{20} ; \qquad B_{is} = B_{oi}-A_{oi}A_{ii}^{-1}B_{ii}$$

$$L_{os} = L_o - \sqrt{\varepsilon_1}\, A_{01}A_{11}^{-1}L_1 - \sqrt{\varepsilon_2}\, A_{02}A_{22}^{-1}L_2 ; \qquad C_{is} = C_{oi} - \frac{1}{\sqrt{\varepsilon_i}}\, \bar{C}_i A_{ii}^{-1}A_{io}$$

$$D_{is} = -\frac{1}{\sqrt{\varepsilon_i}}\, \bar{C}_i A_{ii}^{-1}B_{ii} ; \qquad v_{is} = v_i - \bar{C}_i A_{ii}^{-1}L_i w$$

$$\bar{L}_i = -\bar{C}_i A_{ii}^{-1}L_i ; \qquad V_{is} = V_i + \bar{L}_i W\bar{L}_i'$$

$$Q_{ois} = \bar{Q}_{oi} + (A_{ii}^{-1}A_{io})'\bar{Q}_i(A_{ii}^{-1}A_{io}) ; \qquad Q_{is} = (A_{ii}^{-1}A_{io})'\bar{Q}_i(A_{ii}^{-1}B_{ii})$$

$$R_{is} = R_i + (A_{ii}^{-1}B_{ii})'\bar{Q}_i(A_{ii}^{-1}B_{ii}) ; \qquad L_o^{(i)} = L_o - \sqrt{\varepsilon_j}\, A_{oj}A_{jj}^{-1}L_j$$

$$\tilde{A}_s = A_s - B_{1s}R_{1s}^{-1}Q_{1s}' - B_{2s}R_{2s}^{-1}Q_{2s}' ; \qquad A_o^{(i)} = A_o - A_{oj}A_{jj}^{-1}A_{jo}$$

$$\tilde{A}_{is} = A_s - B_{js}R_{js}^{-1}Q_{js}' ; \quad S_{is} = B_{is}R_{is}^{-1}B_{is}'$$

$$F_s = \begin{bmatrix} \tilde{A}_s - \sum_{i=1}^{2} B_{is}R_{is}^{-1}B_{is}'K_{is} & B_{1s}R_{1s}^{-1}(B_{1s}'K_{1s}+Q_{1s}') & B_{2s}R_{2s}^{-1}(B_{2s}'K_{2s}+Q_{2s}') \\ \tilde{A}_{1s}-F_{1s}-B_{2s}R_{2s}^{-1}B_{2s}'K_{2s} & F_{1s}-G_{1s}C_{1s} & B_{2s}R_{2s}^{-1}(B_{2s}'K_{2s}+Q_{2s}') \\ \tilde{A}_{2s}-F_{2s}-B_{1s}R_{1s}^{-1}B_{1s}'K_{1s} & B_{1s}R_{1s}^{-1}(B_{1s}'K_{1s}+Q_{1s}') & F_{2s}-G_{2s}C_{2s} \end{bmatrix}$$

$$B_s = \begin{bmatrix} -L_{os} & 0 & 0 \\ -L_{os} & G_{1s} & 0 \\ -L_{os} & 0 & G_{2s} \end{bmatrix} ; \qquad \Theta_s = \begin{bmatrix} W & -W\bar{L}_1' & -W\bar{L}_2' \\ -\bar{L}_1 W & V_{1s} & \bar{L}_1 W\bar{L}_2' \\ -\bar{L}_2 W & \bar{L}_2 W\bar{L}_1' & V_{2s} \end{bmatrix}$$

$$F_i = \begin{bmatrix} F_{00}^{(i)}(\epsilon) & F_{01}^{(i)}(\epsilon) & F_{02}^{(i)}(\epsilon) \\ \frac{1}{\epsilon_1}F_{10}^{(i)}(\epsilon) & \frac{1}{\epsilon_1}F_{11}^{(i)}(\epsilon) & \frac{\epsilon_{11}}{\epsilon_1}F_{12}^{(i)}(\epsilon) \\ \frac{1}{\epsilon_2}F_{20}^{(i)}(\epsilon) & \frac{\epsilon_{22}}{\epsilon_2}F_{21}^{(i)}(\epsilon) & \frac{1}{\epsilon_2}F_{22}^{(i)}(\epsilon) \end{bmatrix} ; \quad G_i = \begin{bmatrix} G_{i0}(\epsilon) \\ \frac{1}{\sqrt{\epsilon_1}}\, G_{i1}(\epsilon) \\ \frac{1}{\sqrt{\epsilon_2}}\, G_{i2}(\epsilon) \end{bmatrix}$$

$$\bar{A}_{00} = \text{block diag}[F_{00}^{(1)} - S_{01}K_{00}^{(1)} - \tilde{S}_{01}K_{01}^{(1)'},\; F_{00}^{(2)} - S_{02}K_{00}^{(2)} - \tilde{S}_{02}K_{02}^{(2)'}]$$

$$\bar{A}_{01} = \text{block diag}[F_{01}^{(1)} - \epsilon_1 S_{01}K_{01}^{(1)} - \tilde{S}_{01}K_{11}^{(1)},\; F_{01}^{(2)} - \epsilon_1 S_{02}K_{01}^{(2)} - \sqrt{\frac{\epsilon_1}{\epsilon_2}}\, \tilde{S}_{02}K_{12}^{(2)}]$$

$$\bar{A}_{02} = \text{block diag}[F_{02}^{(1)} - \epsilon_2 S_{01}K_{02}^{(1)} - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\, \tilde{S}_{01}K_{12}^{(1)},\; F_{02}^{(2)} - \epsilon_2 S_{02}K_{02}^{(2)} - \tilde{S}_{02}K_{22}^{(2)}]$$

$$\bar{A}_{10} = \text{block diag}[F_{10}^{(1)} - \tilde{S}_{01}'K_{00}^{(1)} - S_{11}K_{01}^{(1)'},\; F_{10}^{(2)}]$$

$$\bar{A}_{11} = \text{block diag}[F_{11}^{(1)} - \epsilon_1\tilde{S}_{01}'K_{01}^{(1)} - S_{11}K_{11}^{(1)},\; F_{11}^{(2)}]$$

$$\bar{A}_{12} = \text{block diag}[F_{12}^{(1)} - \frac{\epsilon_2}{\epsilon_{11}}\, \tilde{S}_{01}'K_{02}^{(1)} - \sqrt{\frac{\epsilon_2}{\epsilon_1}} \cdot \frac{1}{\epsilon_{11}}\, S_{11}K_{12}^{(1)},\; F_{12}^{(2)}]$$

$$\bar{A}_{20} = \text{block diag}[F_{20}^{(1)},\; F_{20}^{(2)} - \tilde{S}_{02}'K_{00}^{(2)} - S_{22}K_{02}^{(2)'}]$$

$$\bar{A}_{21} = \text{block diag}[F_{21}^{(1)},\; F_{21}^{(2)} - \frac{\epsilon_1}{\epsilon_{22}}\, \tilde{S}_{02}'K_{01}^{(2)} - \sqrt{\frac{\epsilon_1}{\epsilon_2}} \cdot \frac{1}{\epsilon_{22}}\, S_{22}K_{12}^{(2)}]$$

$$\bar{A}_{22} = \text{block diag}[F_{22}^{(1)},\; F_{22}^{(2)} - \epsilon_2\tilde{S}_{02}'K_{02}^{(2)} - S_{22}K_{22}^{(2)}]$$

$$\bar{G}_i = \text{block diag}[G_{1i},\; G_{2i}];\; i = 0,1,2.$$

$$\nu = [\nu_1' \quad \nu_2']'$$

APPENDIX C

PROOF OF PROPOSITION 4.1

The unique optimal solution to the static team problem defined by (4.11a), (4.14) and (4.12) is given by [36],

$$u_i^*(t) = P_i[y_i - C_i\bar{x}_o] - B_i'S\bar{x}(t); \quad i = 1,2. \tag{C1}$$

$S(t)$ is the nonnegative definite solution of the Riccati equation

$$\dot{S} + A'S + SA - S(B_1B_1' + B_2B_2')S + Q = 0; \quad S(t_f) = Q_f \tag{C2}$$

$$\dot{\bar{x}}(t) = (A - B_1B_1'S - B_2B_2'S)\bar{x}(t); \quad \bar{x}(t_o) = \bar{x}_o \tag{C3}$$

$$P_i = B_i'S_i[\tilde{P}_i - \tilde{L}_j\Sigma_i] - B_i'K_i; \quad i,j = 1,2; \quad i \neq j. \tag{C4}$$

$S_i(t)$ is the nonnegative definite solution of the Riccati equation

$$\dot{S}_i = A'S_i + S_iA - S_iB_iB_i'S_i + Q = 0; \quad S_i(t_f) = Q_f; \quad i=1,2 \tag{C5}$$

and

$$\dot{\tilde{P}}_i = [A - B_iB_i'S_i]\tilde{P}_i + B_iB_i'[K_i + S_i\tilde{L}_j\Sigma_i]; \quad \tilde{P}_i(t_o) = 0; \quad i,j = 1,2; \quad i \neq j \tag{C6}$$

$$\dot{\tilde{L}}_i = A\tilde{L}_i + B_iB_i'S_i[\tilde{P}_i - \tilde{L}_j\Sigma_i]C_i - B_iB_i'K_iC_i; \quad L_i(t_o) = I; \quad i,j = 1,2; \quad i \neq j \tag{C7}$$

$$\dot{K}_i = -(A - B_iB_i'S_i)'K_i - S_iB_jB_j'S_j[\tilde{P}_j - \tilde{L}_i\Sigma_j]C_j\Sigma_i + S_iB_jB_j'K_jC_j\Sigma_i;$$
$$K_i(t_f) = 0; \quad i,j = 1,2; \quad i \neq j \tag{C8}$$

$$\Sigma_i = \Sigma_oC_i'(C_i\Sigma_oC_i' + R_i)^{-1}; \quad i = 1,2. \tag{C9}$$

The minimum value J^* is given by

$$J^* = J(u_1^*, u_2^*) = \bar{x}_o'S(0)\bar{x}_o + tr(\Sigma_oS(0)) + tr\left(\int_{t_o}^{t_f} S(t)FF'dt\right) + J_m \tag{C10}$$

where

$$J_m = tr\int_{t_o}^{t_f}\left[\sum_{i=1}^{2}(\Lambda_o^{(i)'}\Lambda_o^{(i)}\Sigma_o + \Lambda_i^{(1)'}\Lambda_i^{(1)}R_i + \Lambda_i^{(2)'}\Lambda_i^{(2)}R_i + SB_iB_i'SW)\right]dt \tag{C11}$$

$$\Lambda_o^{(i)}(t) = P_iC_i - B_i'S(\tilde{L}_i + \tilde{L}_j) + 3B_i'S\Phi(t,t_o); \quad i,j = 1,2; \quad i \neq j \tag{C12}$$

$$\Lambda_i^{(i)}(t) = P_i' + B_i'SV_i; \quad i = 1,2 \tag{C13}$$

$$\Lambda_i^{(j)}(t) = B_j'SV_i; \quad i,j = 1,2; \quad i \neq j \tag{C14}$$

$$\dot{V}_i = AV_i + B_iB_i'S_i[\tilde{P}_i - \tilde{L}_j\Sigma_i] - B_iB_i'K_i; \quad V_i(t_o) = 0 \quad i,j = 1,2; \quad i \neq j \tag{C15}$$

$$\dot{W} = AW + WA' + FF'; \quad W(t_o) = 0 \tag{C16}$$

$$\dot{\Phi} = A\Phi; \quad \Phi(t_o,t_o) = I. \tag{C17}$$

To prove (a) we express $S(t)$, $S_i(t)$, $\tilde{L}_i(t)$, $K_i(t)$, $\tilde{P}_i(t)$ in partitioned form as

$$S(t) = \begin{bmatrix} S_{00} & \varepsilon_1 S_{01} & \varepsilon_2 S_{02} \\ \varepsilon_1 S_{01}' & \varepsilon_1 S_{11} & \sqrt{\varepsilon_1\varepsilon_2} S_{12} \\ \varepsilon_2 S_{02}' & \sqrt{\varepsilon_1\varepsilon_2} S_{12}' & \varepsilon_2 S_{22} \end{bmatrix}; \quad S_i(t) = \begin{bmatrix} S_{00}^{(i)} & \varepsilon_1 S_{01}^{(i)} & \varepsilon_2 S_{02}^{(i)} \\ \varepsilon_1 S_{01}^{(i)'} & \varepsilon_1 S_{11}^{(i)} & \sqrt{\varepsilon_1\varepsilon_2} S_{12}^{(i)} \\ \varepsilon_2 S_{02}^{(i)'} & \sqrt{\varepsilon_1\varepsilon_2} S_{12}^{(i)'} & \varepsilon_2 S_{22}^{(i)} \end{bmatrix}$$

$$\tilde{L}_i(t) = \begin{bmatrix} \tilde{L}_{00}^{(i)} & \tilde{L}_{01}^{(i)} & \tilde{L}_{02}^{(i)} \\ \tilde{L}_{10}^{(i)} & \tilde{L}_{11}^{(i)} & \tilde{L}_{12}^{(i)} \\ \tilde{L}_{20}^{(i)} & \tilde{L}_{21}^{(i)} & \tilde{L}_{22}^{(i)} \end{bmatrix}, \quad K_i(t) = \begin{bmatrix} K_0^{(i)} \\ \varepsilon_1 K_1^{(i)} \\ \varepsilon_2 K_2^{(i)} \end{bmatrix}, \quad \tilde{P}_i(t) = \begin{bmatrix} \tilde{P}_o^{(i)} \\ \tilde{P}_1^{(i)} \\ \tilde{P}_2^{(i)} \end{bmatrix}.$$

Substituting these forms in (C4), expanding out and neglecting $O(\|\varepsilon\|)$ terms, we obtain

$$P_i = (B_{0i}'S_{00}^{(i)} + B_{ii}'S_{0i}^{(i)'})(\tilde{P}_0^{(i)} - \tilde{L}_{00}^{(j)}\Sigma_{is} - \tilde{L}_{0i}^{(j)}\Sigma_{if}) + B_{ii}'S_{ii}^{(i)}(\tilde{P}_i^{(i)} - \tilde{L}_{i0}^{(j)}\Sigma_{is} - \tilde{L}_{ii}^{(j)}\Sigma_{if})$$

$$-B_{0i}' K_0^{(i)} - B_{ii}' K_i^{(i)} \tag{C18}$$

Substituting the partitioned forms in (C5) and (C6)-(C8) and taking the limit as $\|\varepsilon\| \to 0$ we get,

$$\left.\begin{aligned}
S_{00}^{(i)} &= S_{is} + O(\|\epsilon\|) \\
S_{0i}^{(i)} &= S_{is}\, B_{0i}\, B_{ii}'\, \bar{S}_{if}\, \hat{A}_{ii}^{-1} + O(\|\epsilon\|);\ \hat{A}_{ii} = A_{ii} - B_{ii} B_{ii}' \bar{S}_{if} \\
S_{ii}^{(i)} &= \bar{S}_{if} + O(\|\epsilon\|) \\
\tilde{L}_{00}^{(j)} &= \tilde{L}_{js} + O(\|\epsilon\|) \\
\tilde{L}_{ii}^{(j)} &= \Phi_i(t,t_0) + O(\|\epsilon\|) \\
\tilde{L}_{0i}^{(j)} &= O(\|\epsilon\|) \\
S_{ij}^{(i)} &= O(\|\epsilon\|) \\
\tilde{L}_{i0}^{(j)} &= [G_i - \bar{S}_{if}^{-1}\, \hat{A}_{ii}'^{-1}\, B_{ii}]\, B_{0i}'\, S_{is}\, \tilde{L}_{js} + O(\|\epsilon\|) \\
\tilde{P}_0^{(i)} &= \tilde{P}_{is} + O(\|\epsilon\|) \\
\tilde{P}_i^{(i)} &= [G_i - \bar{S}_{if}^{-1}\, \hat{A}_{ii}'^{-1}\, B_{ii}]\, B_{0i}'\, S_{is}\, \tilde{P}_{is} + O(\|\epsilon\|) \\
K_0^{(i)} &= K_{is} + O(\|\epsilon\|) \\
K_i^{(i)} &= \bar{S}_{if}\, G_i\, B_{0i}'\, K_{is} + O(\|\epsilon\|) \\
&\text{where} \\
G_i &= [\hat{A}_{ii}^{-1}\, B_{ii} + \bar{S}_{if}^{-1}\, \hat{A}_{ii}'^{-1}\, \bar{S}_{if}\, B_{ii} + (\hat{A}_{ii}^{-1}\, B_{ii})(\hat{A}_{ii}^{-1}\, B_{ii})'\, \bar{S}_{if}\, B_{ii}] \\
&\qquad i,j=1,2;\ i \neq j
\end{aligned}\right\} \tag{C19}$$

Substituting (C19) into (C18) and manipulating terms we obtain,

$$P_i = P_{is} - B_{ii}'\, \bar{S}_{if}\, \Psi_{if}\, (t,t_0)\, \Sigma_{if} + O(\|\epsilon\|) \tag{C20}$$

Next, consider the second term of $u_i^*(t)$ from (C1):

$$B_i' S \bar{x}(t) = (B_{0i}' S_{00} + B_{ii}' S_{0i}') \bar{\eta}_0(t) + B_{ii}' S_{ii} \bar{\eta}_i(t) \tag{C21}$$

Substituting the partitioned form of S(t) in (C2) and taking the limit as $\|\epsilon\| \to 0$ we obtain

$$\left.\begin{aligned} S_{00} &= S_s + 0(\|\epsilon\|) \\ S_{0i} &= S_s B_{0i} B_{ii}' \bar{S}_{if} \hat{A}_{ii}^{-1} + 0(\|\epsilon\|) \\ S_{ii} &= \bar{S}_{if} + 0(\|\epsilon\|); \; S_{ij} = 0(\|\epsilon\|) \end{aligned}\right\} \tag{C22}$$

Using (C22) in (C3) and taking the limit as $\|\epsilon\| \to 0$, it can be shown that

$$\bar{\eta}_0(t) = \bar{\eta}_{0s}(t) + 0(\|\epsilon\|) \tag{C23}$$

$$\bar{\eta}_i(t) = \bar{\eta}_{if}(t) + \hat{A}_{ii}^{-1} B_{ii} (B_{0i}' S_{00} + B_{ii}' S_{0i}') \bar{\eta}_{0s}(t) + 0(\|\epsilon\|) \tag{C24}$$

Substituting (C23),(C24) in (C21) and rearranging terms we obtain

$$B_i' S \bar{x}(t) = R_{is}^{-1} B_{0i}' S_s \bar{\eta}_{0s}(t) + B_{ii}' \bar{S}_{if} \bar{\eta}_{if}(t) + 0(\|\epsilon\|) \tag{C25}$$

Therefore, (C25) and (C20) imply

$$u_i^*(t) = u_{im}(t) + 0(\|\epsilon\|); \; i=1,2; \; t \in [t_1 t_2] \subset [t_0, t_f]$$

The reason the above approximation is valid only on a subinterval is because we have neglected the boundary-layer terms.

To prove (b), we need to obtain the limiting expressions for the variables V_i, W and consequently for $\Lambda_0^{(i)}$, $\Lambda_i^{(i)}$ and $\Lambda_i^{(j)}$.

$$\text{Let } V_i = \begin{bmatrix} V_0^{(i)} \\ V_1^{(i)} \\ V_2^{(i)} \end{bmatrix} \text{ and } W = \begin{bmatrix} W_{00} & W_{01} & W_{02} \\ W_{01}' & W_{11} & W_{12} \\ W_{02}' & W_{12}' & W_{22} \end{bmatrix}$$

Substituting the partitioned forms of V_i and W in (C15),(C16) and taking the limit as $\|\epsilon\| \to 0$ we obtain

$$\begin{aligned}
V_0^{(i)} &= V_{is} + 0(\|\epsilon\|) \\
V_i^{(i)} &= [\Psi_{if}(t,t_0) - \Phi_i(t,t_0)]\,\Sigma_{if} + [G_i - \bar{S}_{if}^{-1}\hat{A}_{ii}'^{-1}\,B_{ii}]B_{0i}'S_sV_{is} \\
&\qquad + 0(\|\epsilon\|) \\
V_j^{(i)} &= \Phi_j(t,t_0) + (\|\epsilon\|) \\
W_{00} &= W_s + 0(\|\epsilon\|) \\
W_{ii} &= \tilde{W}_i + 0(\|\epsilon\|) \\
W_{0i} &= W_{ij} = 0(\|\epsilon\|)
\end{aligned} \tag{C26}$$

Substituting (C19), (C20), (C22) and (C26) in (C12)-(C17) and manipulating terms results in

$$\begin{aligned}
\Lambda_0^{(i)} &= \Lambda_{0s}^{(i)} + \Lambda_{0f}^{(i)} + 0(\|\varepsilon\|) \\
\Lambda_i^{(i)} &= \Lambda_{is}^{(i)} + \Lambda_{if}^{(i)} + 0(\|\varepsilon\|) \\
\Lambda_i^{(j)} &= \Lambda_{is}^{(j)} + 0(\|\varepsilon\|).
\end{aligned} \tag{C27}$$

Using these limiting values in (C10) and (C11) and simplifying the terms gives us the desired result

$$J^* = J_s^* + \bar{J}_{1f}^* + \bar{J}_{2f}^* + O(\|\varepsilon\|).$$

Neglecting the boundary-layer terms does not affect the approximation in the cost because their contribution to the cost is $O(\|\varepsilon\|)$.

APPENDIX D

MODEL VARIABLES OF THE POWER EXAMPLE OF CHAPTER 6

x_1, x_{12} = valve position displacement in first thermal unit of area 1 and 2.

x_2, x_{13} = power output displacement of HP turbine in first thermal unit of area 1 and 2.

x_3, x_{14} = power output displacement of IP turbine in first thermal unit of area 1 and 2.

x_4, x_{15} = power output displacement of LP turbine in first thermal unit of area 1 and 2.

x_5, x_{16} = valve position displacement in second thermal unit of area 1 and 2.

x_6, x_{17} = power output displacement of HP turbine in second thermal unit of area 1 and 2.

x_7, x_{18} = power output displacement of IP turbine in second thermal unit of area 1 and 2.

x_8, x_{19} = power output displacement of LP turbine in second thermal unit of area 1 and 2.

x_9, x_{11} = frequency deviation of area 1 and 2.

x_{10} = tie-line power flow connecting area 1 and 2.

$u_1^{(1)}, u_1^{(2)}$ = set-point adjustment of first thermal unit in area 1 and 2.

$u_2^{(1)}, u_2^{(2)}$ = set-point adjustment of second thermal unit in area 1 and 2.

$y_1^{(1)}, y_2^{(2)}$ = frequency deviation of area 1 and 2.

$y_2^{(1)}, y_1^{(2)}$ = tie-line power flow of area 1 and 2.

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